MATH 247 Calculus 3, Exercises for Chapter 5

1: (a) Let $(u,v)=f(t)=\big(\cos t+2, 2\sin t-1\big)$ and let $(x,y)=g(u,v)=\left(\frac{u}{v},\frac{v}{u}\right)$. Use the Chain Rule to find the tangent vector to the curve $r(t)=g(f(t))$ at the point where $t=\frac{\pi}{2}$.

(b) Let $u=f(x,y,z)=4\,x\tan^{-1}\left(\frac{y}{z}\right)$ where $(x,y,z)=g(s,t)=\left(s^3+t,\sqrt{s}\,t,\frac{t}{s}\right)$. Use the Chain Rule to find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$ when $(s,t)=(1,-2)$.

2: (a) Let $u=f(x,y,z)=(x+y)e^{y^2+z}$. Find $\nabla f(1,2,-4)$, then find the equation of the tangent plane at $(1,2,-4)$ to the surface $f(x,y,z)=3$, and find the directional derivative $D_uf(1,2,-4)$ where $u=\frac{1}{7}(2,-3,6)$.

(b) Let $f(x,y)=x^2y-y^3$. Find $\nabla f(3,-1)$, then for each of the values $m=0,6,6\sqrt{2}$ and 10, find a unit vector u, if one exists, such that $D_uf(3,-1)=m$.

3: A boy is standing at the point $(5,10,2)$ on a hill in the shape of the surface

$$z=\frac{600}{100+4x^2+y^2}$$

(where x, y and z are in meters).

(a) Sketch the surface.

(b) At the point where the boy is standing, in which direction is the slope steepest?

(c) If the boy walks southeast, then will he be ascending or descending?

(d) If the boy walks in the direction of steepest slope, then at what angle (from the horizontal) will he be climbing?

4: For each of the following fuctions $f:\mathbf{R}^2\to\mathbf{R}$, determine where f is continuous and where f is differentiable.

(a) $f(x,y)=(x^2y^2)^{1/3}$.

(b) $f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^4} & \text{, if } (x,y)\neq(0,0)\\ 0 & \text{, if } (x,y)=(0,0)\end{cases}$

(c) $f(x,y)=\begin{cases}\dfrac{xy^3}{x^2+y^4} & \text{, if } (x,y)\neq(0,0)\\ 0 & \text{, if } (x,y)=(0,0)\end{cases}$

5: (a) For $x\in\mathbf{R}^3$, $y\in\mathbf{R}^2$ and $z\in\mathbf{R}^2$, define $f:\mathbf{R}^5\to\mathbf{R}^2$, written as $z=f(x,y)$, by

$$\begin{pmatrix}z_1\\ z_2\end{pmatrix}=\begin{pmatrix}f_1(x,y)\\ f_2(x,y)\end{pmatrix}=\begin{pmatrix}x_1y_2-4x_2+2e^{y_1}+3\\ 2x_1-x_3+y_2\cos y_1-6y_1\end{pmatrix}.$$

Note that for $a=(3,2,7)$ and $b=(0,1)$ we have $f(a,b)=(0,0)$. Find $Df(a,b)$, explain why near the point (a,b) the null set $\text{Null}(f)$ is locally equal to the graph of a smooth function $g:U\subseteq\mathbf{R}^3\to\mathbf{R}^2$ with $g(a)=b$, and calculate $Dg(a)$.

(b) Let X be the set of all $(a,b,c)\in\mathbf{R}^3$ such that the polynomial $f(t)=t^3+at^2+bt+c$ has a triple root and let Y be the set of $(a,b,c)\in\mathbf{R}^3$ such that $f(t)=t^3+at^2+bt+c$ has a multiple root (that is a double or triple root). Find a parametric equation for X and a parametric equation for Y and show that near every point $(a,b,c)\in Y\setminus X$, the set Y is locally equal to the graph of a smooth function $z=z(x,y)$. As an optional additional exercise, use a computer to sketch the sets X and Y.