MATH 247 Calculus 3, Solutions to the Exercises for Chapter 5

1: (a) Let $(u,v)=f(t)=\big(\cos t+2, 2\sin t-1\big)$ and let $(x,y)=g(u,v)=\left(\frac{u}{v},\frac{v}{u}\right)$. Use the Chain Rule to find the tangent vector to the curve $r(t)=g(f(t))$ at the point where $t=\frac{\pi}{2}$.

Solution: We express the solution in two ways; with and without matrix notation. First we express the solution without matrix notation. We use the Chain Rule in the form

$$\begin{aligned}\frac{dx}{dt}&=\frac{\partial x}{\partial u}\,\frac{du}{dt}+\frac{\partial x}{\partial v}\,\frac{dv}{dt}\\ \frac{dy}{dt}&=\frac{\partial y}{\partial u}\,\frac{du}{dt}+\frac{\partial y}{\partial v}\,\frac{dv}{dt}\end{aligned}$$

We have

$$\frac{\partial x}{\partial u}=\frac{1}{v}\,,\ \frac{\partial x}{\partial v}=-\frac{u}{v^2}\,,\ \frac{\partial y}{\partial u}=-\frac{v}{u^2}\,,\ \frac{\partial y}{\partial v}=\frac{1}{u}\,,\ \frac{du}{dt}=-\sin t\ ,\ \text{and}\ \frac{dv}{dt}=2\cos t\,.$$

When $t=\frac{\pi}{2}$ we have $u=\cos t+2=2$ and $v=2\sin t-1=1$, and so

$$\frac{\partial x}{\partial u}=1\,,\ \frac{\partial x}{\partial v}=-2\,,\ \frac{\partial y}{\partial u}=-\frac{1}{4}\,,\ \frac{\partial y}{\partial v}=\frac{1}{2}\,,\ \frac{du}{dt}=-1\ ,\ \text{and}\ \frac{dv}{dt}=0\,.$$

Put all these values into the two formulas given by the Chain Rule to get

$$\begin{aligned}\frac{dx}{dt}&=\frac{\partial x}{\partial u}\,\frac{du}{dt}+\frac{\partial x}{\partial v}\,\frac{dv}{dt}=(1)(-1)+(-2)(0)=-1\\ \frac{dy}{dt}&=\frac{\partial y}{\partial u}\,\frac{du}{dt}+\frac{\partial y}{\partial v}\,\frac{dv}{dt}=\big(-\tfrac{1}{4}\big)(-1)+\big(\tfrac{1}{2}\big)\,(0)=\tfrac{1}{4}\,.\end{aligned}$$

Thus the tangent vector is $r'\left(\frac{\pi}{4}\right)=\left(x'\left(\frac{\pi}{4}\right),y'\left(\frac{\pi}{4}\right)\right)=\left(-1,\frac{1}{4}\right)$.

Here is the same solution in matrix notation. By the Chain Rule, we have $r'(t)=Dg(f(t))\,f'(t)$, where

$$r'(t)=\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{pmatrix}\ ,\ f'(t)=\begin{pmatrix}\frac{du}{dt}\\ \frac{dv}{dt}\end{pmatrix}=\begin{pmatrix}-\sin t\\ 2\cos t\end{pmatrix}\ ,\ \text{and}\ Dg(u,v)=\begin{pmatrix}\frac{\partial x}{\partial u}&\frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}\end{pmatrix}=\begin{pmatrix}\frac{1}{v}&-\frac{u}{v^2}\\ -\frac{v}{u^2}&\frac{1}{u}\end{pmatrix}$$

When $t=\frac{\pi}{2}$ we have $(u,v)=f\left(\frac{\pi}{2}\right)=(2,1)$, and $f'\left(\frac{\pi}{2}\right)=\begin{pmatrix}-1\\0\end{pmatrix}$ and $Dg(2,1)=\begin{pmatrix}1&-2\\-\frac{1}{4}&\frac{1}{2}\end{pmatrix}$ so the tangent vector at $t=\frac{\pi}{2}$ is

$$r'\left(\tfrac{\pi}{2}\right)=Dg(2,1)\ f'\left(\tfrac{\pi}{2}\right)=\begin{pmatrix}1&-2\\-\frac{1}{4}&\frac{1}{2}\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix}=\begin{pmatrix}-1\\ \frac{1}{4}\end{pmatrix}$$

(b) Let $u = f(x,y,z) = 4\,x\tan^{-1}\left(\frac{y}{z}\right)$ where $(x,y,z) = g(s,t) = \left(s^3+t, \sqrt{s}\,t, \frac{t}{s}\right)$. Use the Chain Rule to find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$ when $(s,t) = (1,-2)$.

Solution: First we give a solution which does not use matrix notation. Note that when $(s,t) = (1,-2)$ we have $(x,y,z) = (-1,-2,-2)$, and at this point we have

$$\frac{\partial u}{\partial x} = 4\tan^{-1}\frac{y}{z} = \pi\,,\ \ \frac{\partial u}{\partial y} = \frac{4x}{1+(y/z)^2}\cdot\frac{1}{z} = 1\,,\ \ \frac{\partial u}{\partial z} = \frac{4x}{1+\left(\frac{y}{z}\right)^2}\left(-\frac{y}{z^2}\right) = -1$$

$$\frac{\partial x}{\partial s} = 3s^2 = 3\,,\ \ \frac{\partial x}{\partial t} = 1\,,\ \ \frac{\partial y}{\partial s} = \frac{t}{2\sqrt{s}} = -1\,,\ \ \frac{\partial y}{\partial t} = \sqrt{s} = 1\,,\ \ \frac{\partial z}{\partial s} = -\frac{t}{s^2} = 2\,,\ \ \frac{\partial z}{\partial t} = \frac{1}{s} = 1$$

and so

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} = (\pi)(3) + (1)(-1) + (-1)(2) = 3\pi - 3\ \text{, and}$$
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = (\pi)(1) + (1)(1) + (-1)(1) = \pi$$

Here is the same solution, using matrix notation. Write $u = h(s,t) = f\big(g(s,t)\big)$. By the Chain Rule, we have $Dh(s,t) = Df\big(g(s,t)\big)Dg(s,t)$. When $(s,t) = (1,-2)$ we have $(x,y,z) = h(s,t) = (-1,-2,-2)$, and at this point

$$\begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \end{pmatrix} = Dh(s,t) = Df(x,y,z)\cdot Dg(s,t) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix}$$
$$= \left(4\tan^{-1}\frac{y}{z}\ ,\ \frac{4x(1/z)}{1+(y/z)^2}\ ,\ \frac{4x(-y/z^2)}{1+(y/z)^2}\right) \begin{pmatrix} 3s^2 & 1 \\ \frac{1}{2\sqrt{t}} & \frac{1}{2\sqrt{s}} \\ -\frac{t}{s^2} & \frac{1}{s} \end{pmatrix}$$
$$= \big(\pi\ ,\ 1\ ,\ -1\big)\begin{pmatrix} 3 & 1 \\ -1 & 1 \\ 2 & 1 \end{pmatrix} = \big(3\pi - 3\ ,\ \pi\big).$$

2: (a) Let $u = f(x,y,z) = (x+y)e^{y^2+z}$. Find $\nabla f(1,2,-4)$, then find the equation of the tangent plane at $(1,2,-4)$ to the surface $f(x,y,z) = 3$, and find the directional derivative $D_u f(1,2,-4)$ where $u = \frac{1}{7}(2,-3,6)$.

Solution: We have $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = e^{y^2+z}\big(1, 1+(x+y)(2y), (x+y)\big)$, so $\nabla f(1,2,-4) = (1,13,3)$. The gradient $(1,13,3)$ is a normal vector, so the equation is of the form $x + 13y + 3z = c$, and by putting in $(x,y,z) = (1,2,-4)$, we find that $c = 15$. Thus the equation is $x + 13y + 3z = 15$. Finally, the directional derivative is $D_u f(1,2,-4) = \nabla f(1,2,-4) \cdot u = \frac{1}{7}\,(1,13,3) \cdot (2,-3,6) = -\frac{19}{7}$.

(b) Let $f(x,y) = x^2y - y^3$. Find $\nabla f(3,-1)$, then for each of the values $m = 0, 6, 6\sqrt{2}$ and 10, find a unit vector u, if one exists, such that $D_u f(3,-1) = m$.

Solution: $\nabla f(x,y) = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right) = (2xy\,,\, x^2 - 3y^2)$ and so $\nabla f(3,-1) = (6,6)$. For each value of m, we need to find a vector $u = (a,b)$ with $a^2 + b^2 = 1$ such that $m = D_u f(3,-1) = \nabla f(3,-1) \cdot u = (6,6) \cdot (a,b) = 6a + 6b$, thus we need to solve the two equations $a^2 + b^2 = 1 \;\; (1)$ and $a + b = \frac{1}{6}\,m \;\; (2)$.

When $m = 0$, equation (2) becomes $a + b = 0$ so that we have $b = -a$. Put $b = -a$ into equation (1) to get $a^2 + (-a)^2 = 1 \Longrightarrow 2a^2 = 1 \Longrightarrow a^2 = \frac{1}{2} \Longrightarrow a = \pm\frac{\sqrt{2}}{2}$. Since $b = -a$, we obtain $(a,b) = \pm\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

When $m = 6$, equation (2) becomes $a + b = 1$ so that we have $b = 1 - a$. Put this into equation (1) to get $a^2 + (1-a)^2 = 1 \Longrightarrow a^2 + 1 - 2a + a^2 = 1 \Longrightarrow 2a^2 - 2a = 0 \Longrightarrow 2a(a-1) = 0 \Longrightarrow a = 0$ or $a = 1$. Since $b = 1 - a$, we obtain $(a,b) = (0,1)$ or $(1,0)$.

When $m = 6\sqrt{2}$, equation (2) becomes $a + b = \sqrt{2}$ so that $b = \sqrt{2} - a$. Put this into equation (1) to get $a^2 + (\sqrt{2} - a)^2 = 1 \Longrightarrow a^2 + 2 - 2\sqrt{2}a + a^2 = 1 \Longrightarrow 2a^2 - 2\sqrt{2}a + 1 = 0 \Longrightarrow 2\big(a - \frac{1}{\sqrt{2}}\big)^2 = 0 \Longrightarrow a = \frac{1}{\sqrt{2}}$. Since $b = \sqrt{2} - a$, we obtain $(a,b) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Finally, note that since the vector $(a,b) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is in the direction of the gradient vector $(6,6)$, it gives the maximum possible value for the directional derivative. So the maximum possible value for $D_u f(3,-1)$ is equal to $6\sqrt{2}$; there is no unit vector such that $D_u f(3,-1) = 10$.

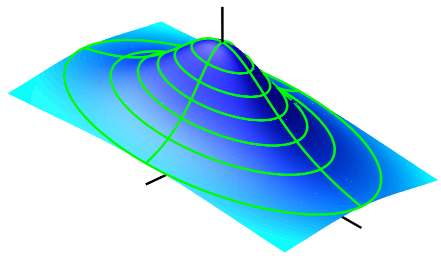
3: A boy is standing at the point $(5, 10, 2)$ on a hill in the shape of the surface

$$z = \frac{600}{100 + 4x^2 + y^2}$$

(where x, y and z are in meters).

(a) Sketch the surface.

Solution: The level set $z = c$ is the ellipse $100 + 4x^2 + y^2 = \frac{600}{c}$, the level set $x = 0$ is the curve $z = \frac{600}{100+y^2}$ and the level set $y = 0$ is the curve $z = \frac{600}{100+4x^2}$. The surface is sketched below.



(b) At the point where the boy is standing, in which direction is the slope steepest?

Solution: Write $z = f(x, y) = \frac{600}{100+4x^2+y^2}$ and $a = (5, 10)$. Then $\nabla f = \left(\frac{-4800\,x}{(100+4x^2+y^2)^2}, \frac{-1200\,y}{(100+4x^2+y^2)^2}\right)$ and so $\nabla f(a) = \left(-\frac{24{,}000}{90{,}000}, -\frac{12{,}000}{90{,}000}\right) = \left(-\frac{4}{15}, -\frac{2}{15}\right) = \frac{2}{15}(-2, -1)$. Thus the slope is the steepest in the direction of the unit vector $\frac{1}{\sqrt{5}}(-2, -1)$.

(c) If the boy walks southeast, then will he be ascending or descending?

Solution: The southeasterly direction is in the direction of the unit vector $v = \frac{1}{\sqrt{2}}(1, -1)$, and the directional derivative in that direction is $D_v f(a) = \frac{2}{15\sqrt{2}}(-2, -1) \cdot (1, -1) = -\frac{\sqrt{2}}{15} < 0$, so the boy would be descending.

(d) If the boy walks in the direction of steepest slope, then at what angle (from the horizontal) will he be climbing?

Solution: If the boy walks in the direction of the unit vector $u = \frac{1}{|\nabla f(a)|}\nabla f(a)$, then the slope in that direction is $D_u f(a) = |\nabla f| = \frac{2}{15}\big|(-2, -1)\big| = \frac{2\sqrt{5}}{15}$, so the angle of ascent is $\theta = \tan^{-1}\frac{2\sqrt{5}}{15} \cong 16.6°$.

4: For each of the following functions $f:\mathbf{R}^2\to\mathbf{R}$, determine where f is continuous and where f is differentiable.

(a) $f(x,y)=(x^2y^2)^{1/3}$.

Solution: Note that f is continuous in $\mathbf{R}^2$ because it is an elementary function, and f is differentiable at all points in the open set $U=\big\{(x,y)\in\mathbf{R}^2\big|xy\neq 0\big\}$ because the restriction $f:U\to\mathbf{R}$ is an open domain elementary function, given by $f=r\circ\big(s\circ(p\cdot q)\big)$ where p, q, r and s are the differentiable functions given by $p(x,y)=x$, $q(x,y)=y$, $s(u)=u^2$ and $r(v)=v^{1/3}$ for $v>0$. It remains to determine whether f is differentiable at points (a,b) with $ab=0$. We claim that f is differentiable at $(0,0)$ but f is not differentiable at points $(a,b)\neq(0,0)$ with $ab=0$. Let $0\neq a\in\mathbf{R}$. If f was differentiable at $(a,0)$ then $\frac{\partial f}{\partial y}(a,0)$ would exist with $\frac{\partial f}{\partial y}(a,0)=g'(0)$ where $g(t)=f(a,t)=(a^2t^2)^{1/3}=a^{2/3}t^{2/3}$, but when $g(t)=a^{2/3}t^{2/3}$ the derivative $g'(0)$ does not exist. Thus f is not differentiable at $(a,0)$ when $a\neq 0$. Similarly, f is not differentiable at $(0,b)$ when $b\neq 0$ because $\frac{\partial f}{\partial x}(0,b)$ does not exist. We claim that f is differentiable at $(0,0)$ with $Df(0,0)=\big(0,0\big)$. Note that $\big|(x,y)-(0,0)\big|=\sqrt{x^2+y^2}$ and recall that for $u,v\in\mathbf{R}$ we have $|uv|\leq\frac{1}{2}(u^2+v^2)\leq(u^2+v^2)$. Let $\epsilon>0$ and choose $\delta=\epsilon^3$. When $\big|(x,y)-(0,0)\big|\leq\delta$, that is when $\sqrt{x^2+y^2}\leq\delta$, we have

$$\left|f(x,y)-f(0,0)-\big(0,0\big)\begin{pmatrix}x-0\\y-0\end{pmatrix}\right|=\big|(x^2y^2)^{1/3}-0-0\big|=|xy|^{2/3}\leq\big(x^2+y^2\big)^{2/3}=(x^2+y^2)^{1/6}(x^2+y^2)^{1/2}$$
$$=\Big(\sqrt{x^2+y^2}\Big)^{1/3}\sqrt{x^2+y^2}\leq\delta^{1/3}\sqrt{x^2+y^2}=\epsilon\,\big|(x,y)-(0,0)\big|$$

and so f is differentiable at $(0,0)$ with $Df(0,0)=\big(0,0\big)$, as claimed.

(b) $f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^4}&,\text{ if }(x,y)\neq(0,0)\\ 0&,\text{ if }(x,y)=(0,0)\end{cases}$

Solution: Note that f is differentiable at all points $(x,y)\neq(0,0)$ because the restriction of f to $\mathbf{R}^2\setminus\{(0,0)\}$ is an open-domain elementary function. We claim that f is also differentiable at $(0,0)$ with $Df(0,0)=\big(0,0\big)$. Let $\epsilon>0$ and choose $\delta=\epsilon$. For $(x,y)\in\mathbf{R}^2$ with $0<\big|(x,y)-(0,0)\big|\leq\delta$, that is with $0<\sqrt{x^2+y^2}\leq\delta$, we have

$$\left|f(x,y)-f(0,0)-\big(0,0\big)\begin{pmatrix}x-0\\y-0\end{pmatrix}\right|=\frac{x^2y^2}{x^2+y^4}\leq\frac{(x^2+y^4)y^2}{x^2+y^4}=y^2\leq x^2+y^2$$
$$=\sqrt{x^2+y^2}\sqrt{x^2+y^2}\leq\delta\sqrt{x^2+y^2}=\epsilon\big|(x,y)-(0,0)\big|$$

so f is indeed differentiable at $(0,0)$. Thus f is differentiable (hence also continuous) at every point $(x,y)\in\mathbf{R}^2$.

(c) $f(x,y)=\begin{cases}\dfrac{xy^3}{x^2+y^4}&,\text{ if }(x,y)\neq(0,0)\\ 0&,\text{ if }(x,y)=(0,0)\end{cases}$

Solution: The function f is continuous and differentiable at all points $(x,y)\neq(0,0)$ because it is equal to an open-domain elementary function on $\mathbf{R}^2\setminus\{(0,0)\}$. Note that f is continuous at $(0,0)$ because for $(x,y)\neq(0,0)$ we have

$$\big|f(x,y)-f(0,0)\big|=\frac{|xy^2||y|}{x^2+y^4}\leq\frac{\frac{1}{2}(x^2+y^4)|y|}{x^2+y^4}=\tfrac{1}{2}|y|\leq\tfrac{1}{2}\sqrt{x^2+y^2}=\tfrac{1}{2}\big|(x,y)-(0,0)\big|.$$

We claim that f is not differentiable at $(0,0)$. For $g_1(t)=f(t,0)$ we have $g_1(t)=0$ for all t (including $t=0$) so $\frac{\partial f}{\partial x}(0,0)=g_1'(0)=0$. For $g_2(t)=f(0,t)$ we have $g_2(t)=0$ for all t so $\frac{\partial f}{\partial y}(0,0)=g_2'(0)=0$. Thus we have $Df(0,0)=\big(0,0\big)$. Let $\alpha(t)=(t^2,t)$ and note that $\alpha'(t)=(2t,1)$ so we have $\alpha(0)=(0,0)$ and $\alpha'(0)=(0,1)$. Let $g(t)=f(\alpha(t))=f(t^2,t)$ and note that $g(t)=\frac{1}{2}t$ for all t (including $t=0$) so we have $g'(t)=\frac{1}{2}$ for all t so, in particular, $g'(0)=\frac{1}{2}$. But if f was differentiable at $(0,0)$ then, by the Chin Rule, we would have $g'(0)=Df(\alpha(0))\alpha'(0)=Df(0,0)(0,1)^T=(0,0)(0,1)^T=0$.

5: (a) For $x \in \mathbf{R}^3$, $y \in \mathbf{R}^2$ and $z \in \mathbf{R}^2$, define $f : \mathbf{R}^5 \to \mathbf{R}^2$, written as $z = f(x, y)$, by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} x_1 y_2 - 4x_2 + 2e^{y_1} + 3 \\ 2x_1 - x_3 + y_2 \cos y_1 - 6y_1 \end{pmatrix}.$$

Note that for $a = (3, 2, 7)$ and $b = (0, 1)$ we have $f(a, b) = (0, 0)$. Find $Df(a, b)$, explain why near the point (a, b) the null set $\text{Null}(f)$ is locally equal to the graph of a smooth function $g : U \subseteq \mathbf{R}^3 \to \mathbf{R}^2$ with $g(a) = b$, and calculate $Dg(a)$.

Solution: We have

$$Df(x,y) = \begin{pmatrix} \frac{\partial z}{\partial z} & \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} & \frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \frac{\partial z_2}{\partial x_3} & \frac{\partial z_2}{\partial y_1} & \frac{\partial z_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} y_2 & -4 & 0 & 2e^{y_1} & x_1 \\ 2 & 0 & -1 & -y_2 \sin y_1 - 6 & \cos y_1 \end{pmatrix}$$

and so

$$Df(a,b) = \begin{pmatrix} \frac{\partial z}{\partial x}(a,b) & \frac{\partial z}{\partial y}(a,b) \end{pmatrix} = \begin{pmatrix} 1 & -4 & 0 & 2 & 3 \\ 2 & 0 & -1 & -6 & 1 \end{pmatrix}.$$

Since the matrix $\frac{\partial z}{\partial y}(a,b) = \begin{pmatrix} 2 & 3 \\ -6 & 1 \end{pmatrix}$ is invertible, the Implicit Function Theorem shows that near the point (a, b), the null set $\text{Null}(f)$ is locally equal to the graph of a smooth function $g : U \subseteq \mathbf{R}^3 \to \mathbf{R}^2$. We have

$$Dg(a) = -\left(\frac{\partial z}{\partial y}(a,b)\right)^{-1} \left(\frac{\partial z}{\partial x}(a,b)\right) = -\frac{1}{20} \begin{pmatrix} 1 & -3 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & -\frac{3}{20} \\ -\frac{1}{2} & \frac{6}{5} & \frac{1}{10} \end{pmatrix}.$$

(b) Let X be the set of all $(a,b,c) \in \mathbf{R}^3$ such that the polynomial $f(t) = t^3 + at^2 + bt + c$ has a triple root and let Y be the set of $(a,b,c) \in \mathbf{R}^3$ such that $f(t) = t^3 + at^2 + bt + c$ has a multiple root (that is a double or triple root). Find a parametric equation for X and a parametric equation for Y and show that near every point $(a,b,c) \in Y \setminus X$, the set Y is locally equal to the graph of a smooth function $z = z(x,y)$. As an optional additional exercise, use a computer to sketch the sets X and Y.

Solution: The monic polynomial with a triple root at $t = u$ is $f(t) = (t-u)^3 = t^3 - 3ut^2 + 3u^2t^2 - u^3$, so X is given parametrically by

$$(x,y,z) = \alpha(u) = (-3u, 3u^2, -u^3).$$

The monic polynomial with double root u and additional root v (possibly with $u = v$) is the polynomial $f(t) = (t-u)^2(t-v) = (t^2 - 2ut + u^2)(t-v) = t^3 - (2u+v)t^2 + (2uv + u^2) - u^2v$, so Y is given parametrically by

$$(x,y,z) = \sigma(u,v) = (-2u - v\,,\, 2uv + u^2, -u^2v)\,.$$

By the Parametric Function Theorem, we know that Range(σ) is locally equal to the graph of a smooth function $z = z(x,y)$ when the top 2×2 submatrix $\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$ of $D\sigma$ is invertible. We have

$$\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} -2 & -1 \\ 2v+2u & 2u \end{pmatrix} = -4u + 2v + 2u = 2(v-u)$$

so the matrix is invertible as long as $u \neq v$. But notice that $u = v$ precisely when $\sigma(u,v) = \alpha(u)$, that is when $\sigma(u,v)$ lies on X.

Let us calculate the function $z = z(x,y)$ explicitly. From $x = -2u - v$ we get $v = -(x+2u)$ then from $y = 2uv + u^2$ we get $y = -2u(x+2u) + u^2 = -3u^2 - 2xu$ so that $3u^2 + 2xu + y = 0$. The Quadratic Formula gives $u = \frac{-2x \pm \sqrt{4x - 12y}}{6} = \frac{-x \pm \sqrt{x^2 - 3y}}{3}$ hence $v = -(x + 2u) = -x + \frac{2x \mp 2\sqrt{x^2-3y}}{3} = \frac{-x \mp 2\sqrt{x^2-3y}}{3}$ (when we use the plus sign for u we must use the minus sign for v and vice versa). Thus the surface is given by

$$\begin{aligned} z = -u^2v &= -\left(\tfrac{-x \pm \sqrt{x^2-3y}}{3}\right)^2\left(\tfrac{-x \mp 2\sqrt{x^2-3y}}{3}\right) = \left(\tfrac{(2x^2-3y) \mp 2x\sqrt{x^2-3y}}{9}\right)\left(\tfrac{x \pm 2\sqrt{x^2-3y}}{3}\right) \\ &= \tfrac{(2x^3 - 3xy) \pm (4x^2 - 6y - 2x^2)\sqrt{x^2-3y} - 4x(x^2-3y)}{27} \\ &= \tfrac{1}{27}\Big((9xy - 2x^3) \pm (2x^2 - 6y)\sqrt{x^2 - 3y}\Big). \end{aligned}$$

Here is a plot which shows that Y is a surface which has a cusp along the twisted cubic curve X.