MATH 247 Calculus 3, Solutions to the Exercises for Chapter 7

1: (a) Find $\displaystyle\int_D xy\,dA$ where D is the region bounded by $x+y=-1$ and $x+y^2=1$.

Solution: Note that $D=\left\{(x,y)\big|-1\le y\le 2\,,\,-1-y\le x\le 1-y^2\right\}$ and so

$$\begin{aligned}\int_D xy\;dA\int_{y=-1}^{2}\int_{x=-1-y}^{1-y^2} xy\;dx\,dy &= \int_{y=-1}^{2}\Big[\tfrac{1}{2}x^2y\Big]_{x=-1-y}^{1-y^2}dy = \int_{-1}^{2}\tfrac{1}{2}y\big((1-y^2)^2-(1+y)^2\big)\,dy\\ &= \int_{y=-1}^{2}\tfrac{1}{2}y\big(1-2y^2+y^4-1-2y-y^2\big)\,dy = \int_{y=-1}^{2}\tfrac{1}{2}y(y^4-3y^2-2y)\,dy\\ &= \int_{y=-1}^{2}\tfrac{1}{12}y^5-\tfrac{3}{2}y^3-y^2\;dy = \Big[\tfrac{1}{12}y^6-\tfrac{3}{8}y^4-\tfrac{1}{3}y^3\Big]_{-1}^{2}\\ &= \big(\tfrac{16}{3}-6-\tfrac{8}{3}\big)-\big(\tfrac{1}{12}-\tfrac{3}{8}+\tfrac{1}{3}\big) = -\tfrac{10}{3}-\tfrac{1}{24} = -\tfrac{81}{24} = -\tfrac{27}{8}.\end{aligned}$$

(b) Find $\displaystyle\int_D x^2+y\;dV$ where D is the tetrahedron bounded by $x+y+z=2$, $z=2$, $x=1$ and $y=x$.

Solution: Although not completely necessary, it helps to sketch the tetrahedron, and to do this it helps to find the vertices. The vertices are at $(1,-1,2)$, $(0,0,2)$, $(1,1,0)$ and $(1,1,2)$ (each vertex can be found by solving three of the four equations $x+y+z=2$, $z=2$, $x=1$ and $y=x$). The region D can be described in several ways, for example we have

$$\begin{aligned}D&=\left\{(x,y,z)\big|0\le x\le 1\,,\,-x\le y\le x\,,\,2-x-y\le z\le 2\right\}\\ &=\left\{(x,y,z)\big|0\le x\le 1\,,\,2-2x\le z\le 2\,,\,2-x-z\le y\le x\right\}\\ &=\left\{(x,y,z)\big|0\le z\le 2\,,\,1-\tfrac{1}{2}z\le x\le 1\,,\,2-x-z\le y\le x\right\}.\end{aligned}$$

Using the first of the above three descriptions of D we have

$$\begin{aligned}\int_D x^2+y\;dV &= \int_{x=0}^{1}\int_{y=-x}^{x}\int_{z=2-x-y}^{2} x^2+y\;dz\,dy\,dx = \int_{x=0}^{1}\int_{y=-x}^{x}\Big[(x^2+y)z\Big]_{z=2-x-y}^{2}dy\,dx\\ &\int_{x=0}^{1}\int_{y=-x}^{x}(x^2+y)\big(2-(2-x-y)\big)dy\,dx = \int_{x=0}^{1}\int_{y=-x}^{x}(x^2+y)(x+y)dy\,dx\\ &= \int_{x=0}^{1}\int_{y=-x}^{x}x^3+(x^2+x)y+y^2\;dy\,dx = \int_{x=0}^{1}\Big[x^3y+\tfrac{1}{2}(x^2+x)y^2+\tfrac{1}{3}y^3\Big]_{y=-x}^{x}dx\\ &= \int_{x=0}^{1}2x^4+\tfrac{2}{3}x^3dx = \Big[\tfrac{2}{5}x^5+\tfrac{1}{6}x^4\Big]_0^1 = \tfrac{2}{5}+\tfrac{1}{6} = \tfrac{17}{30}.\end{aligned}$$

2: (a) Find $\int_D e^{x-y}\,dA$ where D is the parallelogram with vertices at $(1,1)$, $(3,2)$, $(4,5)$ and $(2,4)$.

Solution: We could solve this problem by cutting the parallelogram D into the three regions

$$\begin{aligned} D =& \big\{(x,y)\big|1\le x\le 2\,,\ \tfrac{x+1}{2}\le y\le 3x-2\big\}\cup\big\{(x,y)\big|2\le x\le 3\,,\ \tfrac{x+1}{2}\le y\le \tfrac{x+6}{2}\big\}\\ &\cup\big\{(x,y)\big|3\le x\le 4\,,\ 3x-7\le y\le \tfrac{x+6}{2}\big\}\end{aligned}$$

but instead we will make use of a change in coordinates. The affine map $g:\mathbf{R}^2\to\mathbf{R}^2$ given by

$$\begin{pmatrix}x\\y\end{pmatrix} = g(u,v) = \begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}2&1\\1&3\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$$

sends the square $C=\big\{(u,v)\big|0\le u\le 1\,,\ 0\le v\le 1\big\}$ to the parallelogram D. Let $f(x,y)=e^{x-y}$. Then

$$f(g(u,v)) = f\big(1+2u+v\,,\ 1+u+3v\big) = e^{(1+2u+v)-(1+u+3v)} = e^{u-2v}.$$

so we have

$$\begin{aligned}\iint_D e^{x-y}\,dx\,dy &= \iint_C f(g(u,v))\cdot\det g'(u,v)\,du\,dv = \int_{u=0}^1\int_{v=0}^1 e^{u-2v}\cdot 5\,dv\,du = 5\int_{u=0}^1\Big[-\tfrac{1}{2}e^{u-2v}\Big]_{v=0}^1 du\\ &= -\tfrac{5}{2}\int_0^1 e^{u-2}-e^u\,du = \tfrac{5}{2}\Big[e^u-e^{u-2}\Big]_0^1 = \tfrac{5}{2}\big(e-e^{-1}-1+e^{-2}\big).\end{aligned}$$

(b) Find $\int_D \sqrt{y^3+z}\,dV$ where $D=\big\{(x,y,z)\,\big|\,0\le x\le 1, \sqrt{x}\le y\le 1, y^3\le z\le 1\big\}$.

Solution: Solving the integral with the variables in the given order is challenging, but note that we also have

$$\begin{aligned} D &= \big\{(x,y,z)\,\big|\,0\le x\le 1, \sqrt{x}\le y\le 1, y^3\le z\le 1\big\} = \big\{(x,y,z)\,\big|\,0\le y\le 1, 0\le x\le y^2, y^3\le z\le 1\big\}\\ &= \big\{(x,y,z)\,\big|\,0\le y\le 1\,, y^3\le z\le 1\,, 0\le x\le y^2\big\} = \big\{(x,y,z)\,\big|\,0\le z\le 1, 0\le y\le \sqrt[3]{z}, 0\le x\le y^2\big\}.\end{aligned}$$

Using the last of the above descriptions of D we have

$$\begin{aligned}\int_D \sqrt{y^3+z}\,dV &= \int_{z=0}^1\int_{y=0}^{\sqrt[3]{z}}\int_{x=0}^{y^2}\sqrt{y^3+z}\,dx\,dy\,dz = \int_{z=0}^1\int_{y=0}^{\sqrt[3]{z}} y^2\sqrt{y^3+z}\,dy\,dz\\ &= \int_{z=0}^1\Big[\tfrac{2}{9}(y^3+z)^{3/2}\Big]_{y=0}^{\sqrt[3]{z}}dz = \int_{z=0}^1 \tfrac{2}{9}\big((2z)^{3/2}-z^{3/2}\big)\,dz\\ &= \int_{z=0}^1 \tfrac{2}{9}(2^{3/2}-1)\,z^{3/2}\,dz = \tfrac{4}{45}(2^{3/2}-1).\end{aligned}$$

3: (a) Find the volume of the region which lies under the surface $z = e^{x+y}$ and above the triangle in the xy-plane with vertices at $(0,0)$, $(1,1)$ and $(0,2)$.

Solution: The triangle is the set $D = \left\{(x,y)\middle|0 \le x \le 1, x \le y \le 2-x\right\}$, so the volume is

$$V = \iint_D e^{x+y}\, dA = \int_{x=0}^{1}\int_{y=x}^{2-x} e^{x+y}\, dy\, dx = \int_{x=0}^{1}\Big[e^{x+y}\Big]_{y=x}^{2-x} dx$$
$$= \int_0^1 e^2 - e^2x\, dx = \Big[e^2x - \tfrac{1}{2}e^{2x}\Big]_0^1 = e^2 - \tfrac{1}{2}e^2 + \tfrac{1}{2} = \tfrac{1}{2}(e^2+1).$$

(b) Find the volume of the region which lies outside the cylinder $x^2+y^2=1$, inside the cylinder $x^2+y^2=2x$ and inside the sphere $x^2+y^2+z^2=4$.

Solution: The region $D = \left\{(x,y)\middle|x^2+y^2 \ge 1\,,\ x^2+y^2 \le 2x\right\}$ lies in the right half of the plane and can be expressed conveniently using polar coordinates. Write $x = r\cos\theta$ and $y = r\sin\theta$ with $r > 0$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $x+2+y^2 \ge 1 \iff r^2 \ge 1 \iff r \ge 1$ and $x^2+y^2 \le 2x \iff r^2 \le 2r\cos\theta \iff r \le 2\cos\theta$ and we have $1 = r = 2\cos\theta$ when $\theta = \pm\frac{\pi}{3}$. Thus the polar coordinates map $(x,y) = g(r,\theta) = (r\cos\theta, r\sin\theta)$ sends the region $C = \left\{(r,\theta)\middle| -\frac{\pi}{3} \le \theta \le \frac{\pi}{3}\,,\ 1 \le r \le 2\cos\theta\right\}$ to the region D. The required volume is

$$V = 2\iint_D \sqrt{4-x^2-y^2}\, dx\, dx = 2\iint_C \sqrt{4-r^2}\cdot r\, dr\, d\theta = 4\int_{\theta=0}^{\pi/3}\int_{r=1}^{2\cos\theta} r\sqrt{4-r^2}\, dr\, d\theta$$
$$= 4\int_{\theta=0}^{\pi/3}\Big[-\tfrac{1}{3}(4-r^2)^{3/2}\Big]_{r=1}^{2\cos\theta} d\theta = \int_{\theta=0}^{\pi/3} \tfrac{4}{3}\big(3^{3/2} - (4-4\cos^2\theta)^{3/2}\big)\, d\theta$$
$$= \int_{\theta=0}^{\pi/3} \tfrac{4}{3}\big(3\sqrt{3} - 8\sin^3\theta\big)\, d\theta = \int_{\theta=0}^{\pi/3} 4\sqrt{3} - \tfrac{32}{3}(1-\cos^2\theta)\sin\theta\, d\theta$$
$$= \Big[4\sqrt{3}\,\theta + \tfrac{32}{3}\cos\theta - \tfrac{32}{9}\cos^3\theta\Big]_{\theta=0}^{\pi/3} = \big(\tfrac{4\sqrt{3}\,\pi}{3} + \tfrac{16}{3} - \tfrac{4}{9}\big) - \big(\tfrac{32}{3} - \tfrac{32}{9}\big) = \tfrac{4\sqrt{3}\,\pi}{3} - \tfrac{20}{9}.$$

4: (a) Find the mass of the solid tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with density given by $f(x,y,z) = 1/(1+x)$.

Solution: The tetrahedron is the set $D = \big\{(x,y,z)\big|0 \le x \le 1, 0 \le y \le 1-x, 0 \le z \le 1-x-y\big\}$, and so the mass is

$$\begin{aligned}
M &= \int_D \frac{1}{1+x}\,dV = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} \frac{1}{1+x}\,dz\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{z}{1+x}\right]_{z=0}^{1-x-y} dy\,dx \\
&= \int_{x=0}^{1}\int_{y=0}^{1-x} \frac{1-x-y}{1+x}\,dy\,dx = \int_{x=0}^{1}\left[\frac{(1-x)\,y - \frac{1}{2}y^2}{1+x}\right]_{y=0}^{1-x} dx = \tfrac{1}{2}\int_{x=0}^{1}\frac{(1-x)^2}{1+x}\,dx \\
&= \tfrac{1}{2}\int_0^1 \frac{x^2-2x+1}{x+1}\,dx = \tfrac{1}{2}\int_0^1 x - 3 + \frac{4}{x+1}\,dx \\
&= \tfrac{1}{2}\left[\tfrac{1}{2}x^2 - 3x + 4\ln(x+1)\right]_0^1 = \tfrac{1}{2}\big(\tfrac{1}{2} - 3 + 4\ln 2\big) = 2\ln 2 - \tfrac{5}{4}.
\end{aligned}$$

(b) Find the mass of the solid which lies inside the cone $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 = 4$ with density given by $f(x,y,z) = 2-z$.

Solution: Let $g : \mathbf{R}^3 \to \mathbf{R}^3$ be the spherical coordinates map given by

$$(x,y,z) = g(r,\phi,\theta) = \big(r\cos\theta\sin\phi\,,\ r\sin\theta\sin\phi\,,\ r\cos\phi\big).$$

Recall that the Jacobian of this map is $\det\big(g(r,\phi,\theta)\big) = r^2\sin\phi$. The given solid lies above the xy-plane, and for $r>0$, $0 \le \phi < \frac{\pi}{2}$ and $0 \le \theta \le 2\pi$ we have

$$\begin{gathered}
z \ge \sqrt{x^2+y^2} \iff r\cos\phi \ge r\sin\phi \iff \tan\phi \le 1 \iff 0 \le \phi \le \tfrac{\pi}{4}, \\
x^2+y^2+z^2 \le 4 \iff r^2 \le 4 \iff 0 \le r \le 2,
\end{gathered}$$

and so the map g sends the region $C = \big\{(r,\phi,\theta)\big|0 \le r \le 2, 0 \le \phi \le \frac{\pi}{4}, 0 \le \theta \le 2\pi\big\}$ to the desired region $D = \big\{(x,y,z)\big|z \le \sqrt{x^2+y^2}\,,\ x^2+y^2+z^2 \le 4\big\}$. We have $f\big(g(r,\phi,\theta)\big) = 2 - r\cos\phi$, and so the mass is

$$\begin{aligned}
M &= \int_D f(x,y,z)\ dV = \iiint_C f\big(g(r,\phi,\theta)\big)\cdot\det\big(g'(r,\phi,\theta)\big)\ dr\,d\phi\,d\theta \\
&= \int_{r=0}^{2}\int_{\phi=0}^{\pi/4}\int_{\theta=0}^{2\pi}(2-r\cos\phi)\cdot r^2\sin\phi\ d\theta\,d\phi\,dr = 2\pi\int_{r=0}^{2}\int_{\phi=0}^{\pi/4} 2r^2\sin\phi - r^3\cos\phi\sin\phi\ d\phi\,dr \\
&= 2\pi\int_{r=0}^{2}\Big[-2r^2\cos\phi - \tfrac{1}{2}r^3\sin^2\phi\Big]_{\phi=0}^{\pi/4} = 2\pi\int_0^2 -\sqrt{2}\,r^2 - \tfrac{1}{4}r^3 + 2r^2\ dr \\
&= 2\pi\Big[\tfrac{1}{3}(2-\sqrt{2})r^3 - \tfrac{1}{16}r^4\Big]_0^2 = 2\pi\big(\tfrac{8}{3}(2-\sqrt{2}) - 1\big) = \tfrac{2\pi}{3}(13 - 8\sqrt{2}).
\end{aligned}$$

5: (a) A cord, carrying an unevenly distributed charge, is wound around the cone $z=\sqrt{x^2+y^2}$ following the curve $(x,y,z)=\alpha(t)=(t\cos t,t\sin t,t)$ with $0\leq t\leq 4$. The charge density (charge per unit length) of the cord at position (x,y,z) is given by $f(x,y,z)=z$. Find the total charge of the cord.

Solution: We have $\alpha'(t)=\big(\cos t-t\sin t\,,\,\sin t+t\cos t\,,\,1\big)$ so that

$$\big|\alpha'(t)\big|^2=(\cos^2t-2t\sin t\cos t+t^2\sin^2t)+(\sin^2t-2t\sin t\cos t+t^2\cos^2t)+1=2+t^2.$$

Using the substitution $u=2+t^2$ so that $du=2t\,dt$, the total charge on the cord is

$$\begin{aligned}Q&=\int_{t=0}^{4}f\big(\alpha(t)\big)\big|\alpha'(t)\big|\,dt=\int_{t=0}^{4}t\sqrt{2+t^2}\,dt=\int_{u=2}^{18}\tfrac12u^{1/2}\,du=\Big[\tfrac13u^{3/2}\Big]_{u=2}^{18}\\&=\tfrac13(18\sqrt{18}-2\sqrt2)=\tfrac13(54\sqrt2-2\sqrt2)=\tfrac{52\sqrt2}{3}.\end{aligned}$$

(b) The surface obtained by revolving the circle $(x-1)^2+z^2=1$ in the xz-plane about the z-axis can be given parametrically by

$$(x,y,z)=\sigma(\theta,\phi)=\Big((1+\cos\phi)\cos\theta\,,\,(1+\cos\phi)\sin\theta\,,\,\sin\phi\Big).$$

with $0\leq\theta\leq2\pi$ and $0\leq\phi\leq2\pi$. Find the mass of this surface given that its density (mass per unit area) at position (x,y,z) is given by $f(x,y,z)=1+z^2$.

Solution: We have

$$\sigma_\theta\times\sigma_\phi=\begin{pmatrix}-(1+\cos\phi)\sin\theta\\-(1+\cos\phi)\cos\theta\\0\end{pmatrix}\times\begin{pmatrix}-\sin\phi\cos\theta\\-\sin\phi\sin\theta\\\cos\phi\end{pmatrix}=\begin{pmatrix}(1+\cos\phi)\cos\phi\cos\theta\\(1+\cos\phi)\cos\phi\sin\theta\\(1+\cos\phi)\sin\phi\end{pmatrix}$$

and so

$$\big|\sigma_\phi\times\sigma_\theta\big|=(1+\cos\phi)\sqrt{\cos^2\phi\cos^2\theta+\cos^2\phi\sin^2\theta+\sin^2\phi}=(1+\cos\phi)\sqrt{\cos^2\phi+\sin^2\phi}=1+\cos\phi$$

and we have

$$f\big(\sigma(\phi,\theta)\big)=1+\sin^2\phi$$

so the mass of the surface is

$$\begin{aligned}M&=\int_\sigma f\,dA=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi}f\big(\sigma(\phi,\theta)\big)\big|\sigma_\phi\times\sigma_\theta\big|d\theta\,d\phi=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi}(1+\sin^2\phi)(1+\cos\phi)\,d\theta\,d\phi\\&=2\pi\int_{\phi=0}^{2\pi}1+\cos\phi+\sin^2\!\phi+\sin^2\!\phi\cos\phi\;d\phi=2\pi\big(2\pi+0+\pi+0\big)=6\pi^2\end{aligned}$$

since $\displaystyle\int_0^{2\pi}1\,d\phi=2\pi$, $\displaystyle\int_0^{2\pi}\cos\phi\;d\phi=\Big[\sin\phi\Big]_0^{2\pi}=0$, $\displaystyle\int_0^{2\pi}\sin^2\!\phi\;d\phi=\int_0^{2\pi}\tfrac12(1-\cos2\phi)\,d\phi=\Big[\tfrac12\phi-\tfrac14\sin2\phi\Big]_0^{2\pi}=\pi$ and $\displaystyle\int_0^{2\pi}\sin^2\!\phi\cos\phi\;d\phi=\Big[\tfrac13\sin^3\phi\Big]_0^{2\pi}=0$.