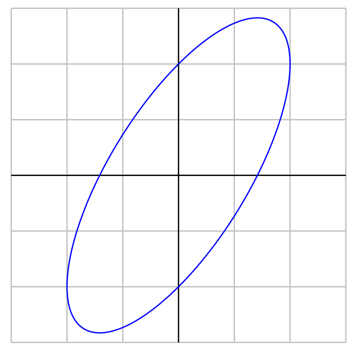
MATH 247 Calculus 3, Solutions to the Midterm Test, Fall 2024

1: Define $f:\mathbb{R}\to\mathbb{R}^2$ by $f(t)=\big(2\sin t\,,\,2\sin t+2\cos t\big)$ and let $C=\text{Range}(f)$.

(a) Sketch the curve C and find the coordinates of all x- and y-intercepts and all points where the tangent line is horizontal or vertical.

Solution: We make a table of values for $0\le t\le\pi$ and plot points below, noting that $f(t+\pi)=-f(t)$. For $(x,y)=f(t)=(2\sin t\,,\,2\sin t+2\cos t)$, the y-intercepts occur when $x=0$, that is when $\sin t=0$, that is when $t=k\pi$ with $k\in\mathbb{Z}$, that is when $(x,y)=\pm(0,2)$. The x-intercepts occur when $y=0$, that is when $\sin t=-\cos t$, that is when $\tan t=-1$, that is when $t=\frac{3\pi}{4}+k\pi$, that is when $(x,y)=\pm(\sqrt{2},0)$. Also, we have $(x',y')=\big(2\cos t\,,\,2\cos t-2\sin t\big)$. The points with vertical tangents occur when $x'=0$, that is $\cos t=0$, that is $t=\frac{\pi}{2}+k\pi$, that is when $(x,y)=\pm(2,2)$, the points with horizontal tangents occur when $y'=0$, that is $\sin t=\cos t$, that is $\tan t=1$, that is $t=\frac{\pi}{4}+k\pi$, that is whe $(x,y)=\pm(\sqrt{2},2\sqrt{2})$.

t	x	y	
0	0	2	y-intercept
$\pi/6$	1	$1+\sqrt{2}$	
$\pi/4$	$\sqrt{2}$	$2\sqrt{2}$	horizontal point
$\pi/3$	$\sqrt{3}$	$1+\sqrt{3}$	
$\pi/2$	2	2	vertical point
$2\pi/3$	$\sqrt{3}$	$\sqrt{3}-1$	
$3\pi/4$	$\sqrt{2}$	0	x-intercept
$5\pi/6$	1	$1-\sqrt{3}$	
π	0	-2	y-intercept



(b) Find (with proof) a polynomial $g(x,y)$ such that $C=\text{Null}(g)$.

Solution: We claim that $\text{Range}(f)=\text{Null}(g)$ where $g(x,y)=x^2+(y-x)^2-4$. Let $(x,y)\in\text{Range}(f)$, say $(x,y)=f(t)=\big(2\sin t\,,\,2\sin t+2\cos t\big)$. Then $g(x,y)=x^2+(y-x)^2-4=(2\sin t)^2+(2\cos t)^2-4=0$ so that $(x,y)\in\text{Null}(g)$. This shows that $\text{Range}(f)\subseteq\text{Null}(g)$. Now let $(x,y)\in\text{Null}(g)$. Then we have $x^2+(y-x)^2=4$, that is $\left(\frac{x}{2}\right)^2+\left(\frac{y-x}{2}\right)^2=1$, and so we can choose $t\in[0,2\pi)$ such that $\sin t=\frac{x}{2}$ and $\cos t=\frac{y-x}{2}$. Then we have $x=2\sin t$ and $y=x+2\cos t=2\sin t+2\cos t$ so that $(x,y)=f(t)\in\text{Range}(f)$. This shows that $\text{Null}(g)\subseteq\text{Range}(f)$.

2: (a) Find an implicit equation for the tangent plane to the surface given by $(x,y,z)=\big(\sqrt{3}\,r\cos\theta\,,\,r^2+r\sin\theta\,,\,2r\big)$ at the point $(3,5,4)$.

Solution: Let $f(r,\theta)=\big(\sqrt{3}\,r\cos\theta,\,r^2+r\sin\theta\,,\,2r\big)$. To get $f(r,\theta)=(3,5,4)$ we need $2r=4$ so that $r=2$, and we need $\cos\theta=\frac{3}{\sqrt{3}r}=\frac{\sqrt{3}}{2}$ and we need $\sin\theta=\frac{5-r^2}{r}=\frac{1}{2}$, so we can take $\theta=\frac{\pi}{6}$. We have

$$Df(r,\theta)=\begin{pmatrix}\sqrt{3}\cos\theta & -\sqrt{3}r\sin\theta\\ 2r+\sin\theta & r\cos\theta\\ 2 & 0\end{pmatrix}\quad\text{so that}\quad Df\big(2,\tfrac{\pi}{6}\big)=\begin{pmatrix}\frac{3}{2} & -\sqrt{3}\\ \frac{9}{2} & \sqrt{3}\\ 2 & 0\end{pmatrix}.$$

The columns of $Df\big(2,\frac{\pi}{6}\big)$ are multiples of $u=(3,2,4)$ and $v=(-1,1,0)$, and $u\times v=(-4,-4,12)=-4w$ where $w=(1,1,-3)$. The tangent plane to the surface at $(3,5,4)$ is the plane through $(3,5,4)$ in the direction of u and v, with normal vector $w=(1,1,-3)$, so it has an equation is of the form $x+y-3z=c$. Since $(3,5,4)$ lies on the plane we have $c=3+5-3\cdot 4=-4$, so the equation is $x+y-3z=-4$.

(b) Find a parametric equation for the tangent line to the curve of intersection of the two paraboloids given by $5z=x^2+y^2$ and $y=x^2+z^2$ at the point $(1,2,1)$.

Solution: Let $g(x,y,z)=\big(u(x,y,z),v(x,y,z)\big)=\big(x^2+y^2-5z\,,\,x^2-y+z^2\big)$ so the given curve is $C=\text{Null}(g)$. We have

$$Dg(x,y,z)=\begin{pmatrix}2x & 2y & -5\\ 2x & -1 & 2z\end{pmatrix}\quad\text{so that}\quad Dg(1,2,1)=\begin{pmatrix}2 & 4 & -5\\ 2 & -1 & 2\end{pmatrix}$$

The row vectors are $\nabla u=(2,4,-5)$ and $\nabla v=(2,-1,2)$. The required tangent line passes through $p=(1,2,1)$ in the direction of $w=\nabla u\times\nabla v=(3,-14,10)$, so it is given parametrically by $(x,y,z)=(1,2,1)+t\,(3,-14,-10)$.

3: (a) Let $f(x,y) = 2x^2y + \frac{y}{x}$. Find every $u \in \mathbb{R}^2$ with $\|u\| = 1$ such that $D_uf(1,2) = 6$.

Solution: We have $f(1,2) = 6$ and we have $Df(x,y) = \big(4xy - \frac{y}{x^2}\,,\, 2x^2 + \frac{1}{x}\big)$ so that $Df(1,2) = (6,3)$. For $u = (a,b)$ we have $D_uf(1,2) = Df(1,2)\binom{a}{b} = 6a + 3b$. To get $D - f(1,2) = 6$ we need $6a + 3b = 6$, that is $2a + b = 3$ (1), and to get $\|u\| = 1$ we need $a^2 + b^2 = 1$ (2). From (1) we get $b = 3 - 2a$, and putting this into (2) gives $a^2 + (3 - 2a)^2 = 1$, that is $5a^2 - 8a + 3 = 0$ so that $(a - 1)(5a - 3) = 0$. Thus $a = 1$ or $a = \frac{3}{5}$ and hence, since $b = 3 - 2a$, we have $u = (a,b) = (1,0)$ or $\big(\frac{3}{5}, \frac{4}{5}\big)$.

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = \big(x^2 + xy\,,\, y^2 - 2xy\big)$. Find $\alpha : \mathbb{R} \to \mathbb{R}^2$ such that for $\beta(t) = f\big(\alpha(t)\big)$ we have $\alpha(0) = (2,1)$, $\alpha(1) = (0,0)$ and $\beta'(0) = (-1,4)$.

Solution: We have $f(2,1) = (5,0)$ and

$$Df(x,y) = \begin{pmatrix} 2x + y & x \\ -2y & 2y - 2x \end{pmatrix} \quad \text{so that} \quad Df(2,1) = \begin{pmatrix} 5 & 2 \\ -2 & -2 \end{pmatrix}.$$

By the Chain Rule, we have $\beta'(0) = Df(\alpha(0))\alpha'(0)$, so, to get $\alpha(0) = (2,1)$ and $\beta'(0) = (-1,4)$ we need

$$\alpha'(0) = Df(1,2)^{-1}\beta'(0) = -\tfrac{1}{6}\begin{pmatrix} 2 & -2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = -\tfrac{1}{6}\begin{pmatrix} -6 \\ 18 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Thus we need to find a function $\alpha = \alpha(t)$ such $\alpha(0) = (2,1)$, $\alpha'(0) = (1,-3)$ and $\alpha(1) = (0,0)$. Writing $\alpha(t) = (x(t), y(t))$, we need $x(0) = 2$, $x'(0) = 1$ and $x(1) = 0$, and we need $y(0) = 1$, $y'(0) = -3$ and $y(1) = 0$. There are many such functions $x(t)$ and $y(t)$ but we can, for example, choose to have x and y be quadratic polynomials. For $x(t) = at^2 + bt + c$, to get $x(0) = 2$ we need $c = 2$, and to get $x'(0) = 1$ we need $b = 1$, and then to get $x(1) = 0$ we need $a + b + c = 0$ so $a = -b - c = -3$, so we can choose $x(t) = -3t^2 + t + 2$. Similarly we can choose $y(t) = 2t^2 - 3t + 1$. Thus one possible choice for α is $\alpha(t) = \big(-3t^2 + t + 2\,,\, 2t^2 - 3t + 1\big)$.

4: (a) Find the mass of the triangle in $\mathbb{R}^2$ with vertices at $(1,1)$, $(3,2)$ and $(0,2)$ with planar density given by $\rho(x,y) = \frac{2x}{y^2}$.

Solution: Note that the given triangle is the set $T = \big\{(x,y) \in \mathbb{R}^2 \,\big|\, 0 \le y \le 2\,,\, 2 - y \le x \le 2y - 1\big\}$, so the mass is

$$\begin{aligned} M &= \int_T \rho = \int_{y=1}^{2}\int_{x=2-y}^{2y-1} \tfrac{2x}{y^2}\,dx\,dy = \int_{y=1}^{2}\Big[\tfrac{x^2}{y^2}\Big]_{x=2-y}^{2y-1} dy = \int_{y=1}^{2} \tfrac{(2y-1)^2-(2-y)^2}{y^2}\,dy \\ &= \int_{y=1}^{2} \tfrac{3y^2-3}{y^2}\,dy = \int_{y=1}^{2} 3 - \tfrac{3}{y^2}\,dy = \Big[3y - \tfrac{3}{y}\Big]_{y=1}^{2} = \big(6 + \tfrac{3}{2}\big) - \big(3 + 3\big) = \tfrac{3}{2}. \end{aligned}$$

(b) Find the volume of the region in $\mathbb{R}^3$ which lies above $2z = x^2 + y^2$ and below $z = x$.

Solution: First let us find the intersection of the paraboloid $2z = x^2 + y^2$ with the plane $z = x$. Put $z = x$ into the equation of the paraboloid to get $2x = x^2 + y^2$, that is $(x - 1)^2 + y^2 = 1$, which is the circle of radius 1 centred at $(1,0)$. It follows that the given region is the given region is the set

$$\begin{aligned} D &= \big\{(x,y,z) \in \mathbb{R}^3 \,\big|\, (x-1)^2 + y^2 \le 1\,,\, \tfrac{x^2+y^2}{2} \le z \le x\big\} \\ &= \big\{(x,y,z) \in \mathbb{R}^3 \,\big|\, 0 \le x \le 2\,,\, -\sqrt{2x - x^2} \le y \le \sqrt{2x - x^2}\,,\, \tfrac{x^2+y^2}{2} \le z \le x\big\}. \end{aligned}$$

Using symmetry, and then using the substitution $\sin\theta = x - 1$ so that $\cos\theta = \sqrt{2x - x^2}$ and $\cos\theta\,d\theta = dx$, the volume is

$$\begin{aligned} V &= 2\int_{x=0}^{2}\int_{y=0}^{\sqrt{2x-x^2}} x - \tfrac{x^2+y^2}{2}\,dy\,dx = \int_{x=0}^{2}\int_{y=0}^{\sqrt{2x-x^2}} 2x - x^2 - y^2\,dx \\ &= \int_{x=0}^{2}\Big[(2x - x^2)y - \tfrac{1}{3}y^3\Big]_{y=0}^{\sqrt{2x-x^2}} dx = \int_{x=0}^{2} \tfrac{2}{3}(2x - x^2)^{3/2}\,dx \\ &= \int_{\theta=-\pi/2}^{\pi/2} \tfrac{2}{3}\cos^4\theta\,d\theta = \tfrac{1}{6}\int_{\theta=-\pi/2}^{\pi/2} (1 + \cos 2\theta)^2\,d\theta \\ &= \tfrac{1}{6}\int_{\theta=-\pi/2}^{\pi/2} 1 + 2\cos\theta + \cos^2\theta\,d\theta = \tfrac{1}{6}\big(\pi + 0 + \tfrac{\pi}{2}\big) = \tfrac{\pi}{4}. \end{aligned}$$

We remark that an alternate solution is obtained using the cylindrical coordinates map g noting that $g(C) = D$ for the set $C = \big\{(r,\theta,z) \in \mathbb{R}^3 \,\big|\, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\,,\, 0 \le r \le 2\cos\theta\,,\, \frac{1}{2}r^2 \le z \le r\cos\theta\big\}$.