PMATH 333 Real Analysis, Assignment 1 Due Mon Sept 23, 11:00 pm

1: (a) Use induction to show that $\sum\limits_{k=2}^{n} \frac{1}{k^3-k} = \frac{(n-1)(n+2)}{4\,n(n+1)}$ for all $n \in \mathbb{Z}$ with $n \geq 2$.

(b) Show that $\sum\limits_{k=1}^{n} k\binom{n}{k} = n\,2^{n-1}$ for all $n \in \mathbb{Z}^+$.

(c) Show that $\sum\limits_{k=1}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}$ for all $n \in \mathbb{Z}^+$.

2: In Appendix 2 (Algebra Lecture Notes), read Definitions 1.1 and 1.2 on page 88, and read Examples 1.33, 1.34 and 1.35 on page 10, then express each of the following statements as formulas in first-order set theory.

(a) x is the union of all the sets which are elements of y (for example, if $y = \{u, v\}$ then $x = u \cup v$).

(b) $x = \{y, y \cup z\}$.

(c) The collection of all one-element sets is not a set (in other words, there does not exist a set whose elements are all of the one-element sets).

3: In Appendix 2 (Algebra Lecture Notes), read the ZFC axioms on page 90, then solve the following problems.

(a) Let u and v be sets. Show (using the ZFC axioms) that $u \cap v$ is a set.

(b) Show that the collection $w = \big\{\{0, 1, 2, \cdots\}, \{1, 2, 3, \cdots\}, \{2, 3, 4, \cdots\}, \cdots\big\}$ is a set. You may assume that $\mathbb{N}$ is a set and that the statements $u = \mathbb{N}$ and $u \in \mathbb{N}$ are expressible as formulas in first-order set theory.

(c) Let F be the formula $\exists w\, \forall z\big(z \in w \leftrightarrow \exists u\, \forall x(x \in z \leftrightarrow \exists y(y \in u \wedge x \in y))\big)$. Determine whether F is a true statement (assuming that the variables represent sets).

4: As in Chapter 1, let R1-R9 be the rules for rings and fields, and let O1-O5 be the rules for ordered fields. Also, let $R0$ be the rule which states that, in a ring R, we have $a \cdot 0 = 0$ for all $a \in R$. For the following problems, provide a step-by-step proof which uses only one rule at each step.

(a) Let F be a field. Using only rules R0-R9, prove that for all $a, b \in F$, if $a \cdot b = 0$ then $a = 0$ or $b = 0$.

(b) Let R be a ring. Using only rules R0-R7, prove that for all $a, b \in R$ if $(a + b) \cdot x = x + b$ for all $x \in R$ then $a = 1$ and $b = 0$.

(c) Let F be an ordered field. Using only rules R0-R9 and O1-O5, prove that for all $a \in F$ we have $0 \leq a \cdot a$.