PMATH 333 Real Analysis, Solutions to Assignment 1

1: (a) Use induction to show that $\sum_{k=2}^{n} \frac{1}{k^3-k} = \frac{(n-1)(n+2)}{4\,n(n+1)}$ for all $n \in \mathbb{Z}$ with $n \geq 2$.

Solution: For $n \geq 2$, let $S_n = \sum_{k=2}^{n} \frac{1}{k^3-k}$ and let $g(n) = \frac{(n-1)(n+2)}{4\,n(n+1)}$. Note that $S_2 = \sum_{k=2}^{2} \frac{1}{k^3-k} = \frac{1}{2^3-2} = \frac{1}{6}$ and $g(2) = \frac{1\cdot 4}{4\cdot 2\cdot 3} = \frac{1}{6}$ so we have $S_2 = g(2)$. Let $n \geq 2$ and suppose, inductively, that $S_n = g(n)$. Then

$$\begin{aligned} S_{n+1} &= \sum_{k=2}^{n+1} \frac{1}{k^3-k} = \sum_{k=2}^{n} \frac{1}{k^3-k} + \frac{1}{(n+1)^3-(n+1)} = S_n + \frac{1}{(n+1)\big((n+1)^2-1\big)} \\ &= g(n) + \frac{1}{(n+1)(n^2+2n)} \text{ by the induction hypothesis} \\ &= \frac{(n-1)(n+2)}{4\,n(n+1)} + \frac{1}{n(n+1)(n+2)} = \frac{(n-1)(n+2)^2+4}{4n(n+1)(n+2)} \\ &= \frac{(n-1)(n^2+4n+4)+4}{4n(n+1)(n+2)} = \frac{n^3+3n^2}{4n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)} = g(n+1), \end{aligned}$$

as required. By induction, we have $S_n = g(n)$ for all $n \geq 2$.

(b) Show that $\sum_{k=1}^{n} k\binom{n}{k} = n\,2^{n-1}$ for all $n \in \mathbb{Z}^+$.

Solution: We provide three solutions. The first solution uses induction. For $n \geq 1$, let $S_n = \sum_{k=1}^{n} k\binom{n}{k}$ and let $g(n) = n\,2^{n-1}$. Note that $S_1 = \sum_{k=1}^{1} k\binom{n}{k} = 1 \cdot \binom{1}{1} = 1$ and $g(1) = 1 \cdot 2^0 = 1$ and so $S_1 = g(1)$. Let $n \geq 1$ and suppose, inductively, that $S_n = g(n)$. Recall that, by the Binomial Theorem, we have $\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$. Also note that for $k \geq 1$, $k\binom{n+1}{k} = k \cdot \frac{(n+1)!}{k!(n+1-k)!} = \frac{(n+1)\;n!}{(k-1)!(n-(k-1))!} = (n+1)\binom{n}{k-1}$. Thus, letting $\ell = k-1$, we have

$$S_{n+1} = \sum_{k=1}^{n+1} k\binom{n+1}{k} = \sum_{k=1}^{n+1} (n+1)\binom{n}{k-1} = (n+1)\sum_{\ell=0}^{n} \binom{n}{\ell} = (n+1)\cdot 2^n = g(n+1)$$

as required. By induction, we have $S_n = g(n)$ for all $n \geq 1$.

For the second solution, we use differentiation. By the Binomial Theorem, we have $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ for all $x \in \mathbb{R}$. Taking the derivative on both sides gives $n\,(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}$. Putting in $x = 1$ gives $n\,2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$, as required.

The third solution uses a counting argument (this is not really a rigorous mathematical proof, but it is a convincing explanation). Let $S = \{1, 2, \cdots, n\}$. We count the number of ways N to choose a non-empty subset $A \subseteq S$ along with an element $a \in A$. On the one hand, for each k with $1 \leq k \leq n$, the number of ways to choose a k-element subset A is equal to $\binom{n}{k}$ and, having chosen A, the number of ways to choose the point $a \in A$ is equal to k, and so we have $N = \sum_{k=1}^{n} k\binom{n}{k}$. On the other hand, the number of ways to choose the point a is equal to n and, having chosen a, the set $A \setminus \{a\}$ can be chosen to be any of the 2^{n-1} subsets of the $(n-1)$-element set $S \setminus \{a\}$, and so we have $N = n\,2^{n-1}$.

(c) Show that $\sum_{k=1}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}$ for all $n \in \mathbb{Z}^+$.

Solution: We provide three solutions. The first solution uses formulas from our first solution to Part (b). We found that $k\binom{n+1}{k} = (n+1)\binom{n}{k-1}$, $\sum_{k=0}^{n}\binom{n}{k} = 2^n$, and $\sum_{k=1}^{n} k\binom{n}{k} = n\,2^{n-1}$. Replacing n by $n-1$ and letting $\ell = k$, we obtain $k\binom{n}{k} = n\binom{n-1}{k-1}$, $\sum_{\ell=0}^{n-1}\binom{n}{\ell} = 2^{n-1}$ and $\sum_{\ell=1}^{n-1}\ell\binom{n-1}{\ell} = (n-1)\,2^{n-2}$. Letting $\ell = k-1$,

$$\begin{aligned}\sum_{k=1}^{n} k^2\binom{n}{k} &= \sum_{k=1}^{n} k\,n\binom{n-1}{k-1} = n\sum_{\ell=0}^{n-1}(\ell+1)\binom{n-1}{\ell} = n\Big(\sum_{\ell=0}^{n-1}\ell\binom{n-1}{\ell} + \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\Big)\\ &= n\Big((n-1)\,2^{n-2} + 2^{n-1}\Big) = n(n+1)2^{n-2}.\end{aligned}$$

The second solution uses differentiation. By the Binomial Theorem, we have $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$ for all $x \in \mathbb{R}$. Differentiate both sides to get $n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}x^{k-1}$. Multiply both sides by x to get $n\,x(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}x^k$. Differentiate again to get $n\big((1+x)^{n-1} + x\cdot(n-1)(1+x)^{n-2}\big) = \sum_{k=1}^{n} k^2\binom{n}{k}x^{k-1}$. Put in $x=1$ to get $\sum_{k=1}^{n} k^2\binom{n}{k} = n\big(2^{n-1} + (n-1)\,2^{n-2}\big) = n(n+1)\,2^{n-2}$.

The third solution uses a counting argument. Let $S = \{1, 2, \cdots, n\}$. We count the number of ways N to choose a nonempty subset $A \subseteq S$ along with an ordered pair $(a,b) \in A^2$. On the one hand, for each $1 \le k \le n$, the number of ways to choose a k-element subset A is equal to $\binom{n}{k}$ and, having chosen A, the number of ways to choose $(a,b) \in A^2$ is equal to k^2, and hence we have $N = \sum_{k=1}^{n} k^2\binom{n}{k}$. On the other hand, the number of ways to choose a pair $(a,b) \in S^2$ with $a \ne b$ is equal to $n(n-1)$ and, having chosen $a, b \in S$, the set $A \setminus \{a,b\}$ can be chosen to be any of the 2^{n-2} subsets of $S \setminus \{a,b\}$, and the number of ways to choose a pair $(a,b) \in S^2$ with $a = b$ is equal to n and, having chosen $a = b \in S$, the set $A \setminus \{a,b\} = A \setminus \{a\}$ can be chosen to be any of the 2^{n-1} subsets of $S \setminus \{a\}$, and so we have $N = n(n-1)\,2^{n-2} + n\,2^{n-1} = n(n+1)\,2^{n-2}$.

2: In Appendix 2 (Algebra Lecture Notes), read Definitions 1.1 and 1.2 on page 88, and read Examples 1.33, 1.34 and 1.35 on page 10, then express each of the following statements as formulas in first-order set theory.

(a) x is the union of all the sets which are elements of y (for example, if $y = \{u, v\}$ then $x = u \cup v$).

Solution: One possible translation is $\forall z\,\big(z \in x \leftrightarrow \exists u(u \in y \wedge z \in u)\big)$.

(b) $x = \{y, y \cup z\}$.

Solution: The statement can be partially translated as $\forall u\,\big(u \in x \leftrightarrow (u = y \vee u = y \cup z)\big)$ and then further partially translated as $\forall u\,\big(u \in x \leftrightarrow (u = y \vee \forall v(v \in u \leftrightarrow v \in y \cup z))\big)$ and then fully translated into the formula

$$\forall u\,\Big(u \in x \leftrightarrow \big(u = y \vee \forall v(v \in u \leftrightarrow (v \in y \vee v \in z))\big)\Big).$$

(c) The collection of all one-element sets is not a set (in other words, there does not exist a set whose elements are all of the one-element sets).

Solution: The (false) statement "x is the set of all sets" can be partially translated as

$$\forall y\,\big(y \in x \leftrightarrow (y \text{ is a one-element set})\big)$$

which can then be translated into the formula

$$\forall y\,\big(y \in x \leftrightarrow \exists z(z \in y \wedge \forall u(u \in y \Longrightarrow u = z))\big).$$

Thus the given statement (namely the true statement that no such set x exists) can be translated into the formula

$$\neg\,\exists x\,\forall y\,\big(y \in x \leftrightarrow \exists z\big(z \in y \wedge \forall u(u \in y \Longrightarrow u = z)\big)\big).$$

3: In Appendix 2 (Algebra Lecture Notes), read the ZFC axioms on page 90, then solve the following problems.

(a) Let u and v be sets. Explain, using the ZFC axioms, why $u \cap v$ is a set.

Solution: The collection $\{u, v\}$ is a set by the Pair Axiom, the collection $u \cup v = \bigcup\{u, v\}$ is a set by the Union Axiom, and so the collection $u \cap v = \big\{x \in u \cup v \,\big|\, x \in u \text{ and } x \in v\big\}$ is a set by a Separation Axiom.

Alternatively, we can argue that $u \cap v = \big\{x \in u \,\big|\, x \in u \text{ and } x \in v\big\}$, which is a set by a Separation Axiom.

(b) Show that the collection $w = \big\{\{0, 1, 2, \cdots\}, \{1, 2, 3, \cdots\}, \{2, 3, 4, \cdots\}, \cdots\big\}$ is a set. You may assume that $\mathbb{N}$ is a set and that the statements $u = \mathbb{N}$ and $u \in \mathbb{N}$ are expressible as formulas in first-order set theory.

Solution: In the class of sets, when $n \in \mathbb{N}$, we have

$$\{n, n+1, n+2, \cdots\} = \mathbb{N} \setminus \{0, 1, \cdots, n-1\} = \mathbb{N} \setminus n = \big\{x \big| x \in \mathbb{N} \wedge \neg\, x \in n\big\} = \big\{x \in \mathbb{N} \big| \neg\, x \in n\big\}$$

which is a subset of $\mathbb{N}$ by a Separation Axiom, and we have

$$u = \{n, n+1, n+2 + \cdots\} \iff u = \mathbb{N} \setminus n \iff \forall x\,\big(x \in u \leftrightarrow (x \in \mathbb{N} \wedge \neg\, x \in n)\big).$$

Let $F(n, u)$ be the formula above on the right. The given collection w is given by

$$w = \big\{u \,\big|\, \exists n \in \mathbb{N}\; F(n, u)\big\}$$

which is a set by a Replacement Axiom, since $\mathbb{N}$ is a set and since the formula $F(n, u)$ has the property that for every set n there is a unique set u such that $F(n, u)$ is true, namely the set $u = \mathbb{N} \setminus n = \big\{x \in \mathbb{N} \big| \neg\, x \in n\big\}$, which is a set by a Separation Axiom.

Alternatively, the given collection w is given by

$$w = \big\{u \in P(\mathbb{N}) \,\big|\, \exists n\, \big(n \in \mathbb{N} \wedge F(n, u)\big)\big\}$$

which is a set by a Separation Axiom, since $P(\mathbb{N})$ is a set by the Power Set Axiom.

(c) Let F be the formula $\exists w\, \forall z\big(z \in w \leftrightarrow \exists u\, \forall x(x \in z \leftrightarrow \exists y(y \in u \wedge x \in y))\big)$. Determine whether F is a true statement (assuming that the variables represent sets).

Solution: In the class of sets, we have

$$\exists y(y \in u \wedge x \in y) \iff x \in \bigcup u$$

$$\forall x(x \in z \leftrightarrow \exists y(y \in u \wedge x \in y)) \iff z = \bigcup u$$

$$\forall z\big(z \in w \leftrightarrow \exists u\, \forall x(x \in z \leftrightarrow \exists y(y \in u \wedge x \in y))\big) \iff w = \big\{z \big| z = \bigcup u \text{ for some set } u\big\}$$

Thus the given formula F states that the class w of all sets of the form $\bigcup u$, for some set u, is a set. We claim that F is false. Suppose, for a contradiction, that the class $w = \big\{z \big| z = \bigcup u \text{ for some set } u\big\}$ is a set. Let v be any set. Note that $v = \bigcup\{v\}$ and so $v = \bigcup u$ for some set u (namely the set $u = \{v\}$) and hence $v \in w$. Since v was arbitrary, we have shown that every set is in w, so w is the class of all sets. Thus w is not a set and so F is false, as claimed.

4: As in Chapter 1, let R1-R9 be the rules for rings and fields, and let O1-O5 be the rules for ordered fields, and let $R0$ be the rule which states that, in a ring R, we have $a \cdot 0 = 0$ for all $a \in R$. For the following problems, provide a step-by-step proof which uses only one rule at each step.

(a) Let F be a field. Using only rules R0-R9, prove that for all $a, b \in F$, if $a \cdot b = 0$ then $a = 0$ or $b = 0$.

Solution: Let $a, b \in F$ and suppose that $a \cdot b = 0$. Suppose that $a \neq 0$. By R9 we can choose $c \in F$ such that $a \cdot c = 1$. Then

$$\begin{aligned} b &= 1 \cdot b \text{ , by R6,} \\ &= (a \cdot c) \cdot b \text{ , since } a \cdot c = 1, \\ &= (c \cdot a) \cdot b \text{ , by R8,} \\ &= c \cdot (a \cdot b) \text{ , by R5,} \\ &= c \cdot 0 \text{ , since } a \cdot b = 0, \\ &= 0 \text{ , by R0.} \end{aligned}$$

(b) Let R be a ring. Using only rules R0-R7, prove that for all $a, b \in R$ if $(a+b)x = x + b$ for all $x \in R$ then $a = 1$ and $b = 0$.

Solution: Let $a, b \in R$. Suppose that $(a+b) \cdot x = x + b$ for all $x \in R$. Then, in particular (taking $x = 0$) we have $(a+b) \cdot 0 = 0 + b$ and so

$$\begin{aligned} 0 &= (a+b) \cdot 0 \text{ , by R0,} \\ &= 0 + b \text{ , as mentioned above,} \\ &= b + 0 \text{ , by R2,} \\ &= b \text{ , by R3.} \end{aligned}$$

This proves that $b = 0$. Since $(a+b) \cdot x = x + b$ for all $x \in R$ it follows in particular, by taking $x = 1$, that $(a+b) \cdot 1 = 1 + b$ and, since $b = 0$, we have $(a+0) \cdot 1 = 1 + 0$. Thus

$$\begin{aligned} a &= a + 0 \text{ , by R3,} \\ &= (a+0) \cdot 1 \text{ , by R6,} \\ &= 1 + 0 \text{ , as mentioned above,} \\ &= 1 \text{ , by R3.} \end{aligned}$$

(c) Let F be an ordered field. Using only rules R0-R9 and O1-O5, prove that for all $a \in F$ we have $0 \leq a \cdot a$.

Solution: Let $a \in F$. By O1, either we have $0 \leq a$ or we have $a \leq 0$. If $0 \leq a$ then we have $0 \leq a \cdot a$ by O5. Suppose that $a \leq 0$. By R4, we can choose $b \in R$ so that $a + b = 0$. Note that

$$\begin{aligned} a \cdot a + a \cdot b &= a \cdot (a + b) \text{ , by R7,} \\ &= a \cdot 0 \text{ , since } a + b = 0, \\ &= 0 \text{ , by R0,} \end{aligned}$$

and hence

$$\begin{aligned} b \cdot b &= b \cdot b + 0 \text{ , by R3,} \\ &= 0 + b \cdot b \text{ , by R2,} \\ &= (a \cdot a + a \cdot b) + b \cdot b \text{ , since } a \cdot a + a \cdot b = 0, \\ &= a \cdot a + (a \cdot b + b \cdot b) \text{ , by R1,} \\ &= a \cdot a + (a + b) \cdot b \text{ , by R7,} \\ &= a \cdot a + 0 \cdot b \text{ , since } a + b = 0, \\ &= a \cdot a + b \cdot 0 \text{ , by R8,} \\ &= a \cdot a + 0 \text{ , by R0,} \\ &= a \cdot a \text{ , by R3.} \end{aligned}$$

Thus we have

$a + b \leq 0 + b$	by O4, since $a \leq 0$
$0 \leq 0 + b$	since $a + b = 0$
$0 \leq b + 0$	by R2
$0 \leq b$	by R3
$0 \leq b \cdot b$	by O5
$0 \leq a \cdot a$	since $b \cdot b = a \cdot a$.