PMATH 333 Real Analysis, Assignment 2 — Due Mon Oct 7, 11:00 pm

1: (a) Let $x_n = \frac{2n+1}{n-1}$ for $n \geq 2$. Use the definition of the limit to show that $\lim_{n\to\infty} x_n = 2$.

(b) Let $x_n = \frac{n}{\sqrt{n+3}}$ for $n \geq 0$. Use the definition of the limit to show that $\lim_{n\to\infty} x_n = \infty$.

(c) Show, from the definition of the limit, that if $x_n \geq 0$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = a$ then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{a}$.

2: (a) Prove that there exist (at least) 3 distinct values of x such that $8x^3 = 6x + 1$.

(b) Let $f : [0, 2] \to \mathbb{R}$ be continuous with $f(0) = f(2)$. Prove that $f(x) = f(x + 1)$ for some $x \in [0, 1]$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that f is bijective (that is, f is injective and surjective).

3: (The Natural Base e) Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ for $n \geq 0$ and let $a_n = \left(1 + \frac{1}{n}\right)^n$ for $n \geq 1$.

(a) Show that $(s_n)_{n\geq 0}$ is increasing and bounded above by 3, and let $e_1 = \lim_{n\to\infty} s_n$.

(b) Use the Binomial Theorem to show that

$$a_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right).$$

(c) Show that $(a_n)_{n\geq 1}$ is increasing with $a_n \leq s_n$ for all $n \geq 1$, and let $e_2 = \lim_{n\to\infty} a_n$.

(d) Show that $e_2 \geq s_n$ for all $n \geq 0$ and hence $e_2 = e_1$.

4: (a) Show that every sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}$ has a monotonic subsequence (that is either $(x_n)_{n\geq 1}$ has an increasing subsequence or $(x_n)_{n\geq 1}$ has a decreasing subsequence). Hint: consider indices n such that $a_n > a_k$ for all $k > n$.

(b) Let $x_n = \frac{n}{\sqrt{2}} - \left\lfloor \frac{n}{\sqrt{2}} \right\rfloor$ for $n \geq 1$. Show that $(x_n)_{n\geq 1}$ has a decreasing subsequence $(x_{n_k})_{k\geq 1}$ with $\lim_{k\to\infty} x_{n_k} = 0$. Hint: consider $(1 + \sqrt{2})^k$ and $(1 - \sqrt{2})^k$.