# PMATH 333 Real Analysis, Solutions to Assignment 2

**1:** (a) Let $x_n = \frac{2n+1}{n-1}$ for $n \geq 2$. Use the definition of the limit to show that $\lim\limits_{n\to\infty} x_n = 2$.

Solution: For $n \geq 2$ and $\epsilon > 0$, we have

$$|x_n - 2| = \left|\tfrac{2n+1}{n-1} - 2\right| = \left|\tfrac{2n+1-2n+2}{n-1}\right| = \tfrac{3}{n-1}$$

and

$$\tfrac{3}{n-1} \leq \epsilon \leftrightarrow \tfrac{n-1}{3} \geq \tfrac{1}{\epsilon} \leftrightarrow n-1 \geq 3\epsilon \leftrightarrow n \geq 3\epsilon + 1.$$

Let $\epsilon > 0$. Choose $m \in \mathbb{Z}$ with $m \geq 3\epsilon + 1$. For $n \in \mathbb{Z}_{\geq 2}$ with $n \geq m$ we have $n \geq m \geq 3\epsilon + 1$ and hence, as shown above, $|x_n - 2| = \frac{3}{n-1} \leq \epsilon$.

(b) Let $x_n = \frac{n}{\sqrt{n+3}}$ for $n \geq 0$. Use the definition of the limit to show that $\lim\limits_{n\to\infty} x_n = \infty$.

Solution: First note that for $n \geq 1$ we have $n + 3 \leq n + 3n = 4n$ and so

$$x_n = \tfrac{n}{\sqrt{n+3}} \geq \tfrac{n}{\sqrt{4n}} = \tfrac{\sqrt{n}}{2}.$$

Let $r \in \mathbb{R}$. Choose $m \in \mathbb{Z}$ with $m \geq 4r^2$. Then for $n \geq m$ we have $n \geq 4r^2$ and so

$$x_n \geq \tfrac{\sqrt{n}}{2} \geq \tfrac{\sqrt{4r^2}}{2} = \tfrac{2|r|}{2} = |r| \geq r.$$

(c) Show that if $x_n \geq 0$ for all $n \geq p$ and $\lim\limits_{n\to\infty} x_n = a \geq 0$ then $\lim\limits_{n\to\infty} \sqrt{x_n} = \sqrt{a}$.

Solution: Consider the case that $a = 0$. Suppose that $x_n \geq 0$ for all $n$ and that $x_n \to 0$. Let $\epsilon > 0$. Since $x_n \to 0$, we can choose $m \in \mathbb{Z}$ so that $n \geq m \Longrightarrow |x_n - 0| \leq \epsilon^2$. Then for $n \geq m$ we have $0 \leq x_n \leq \epsilon^2$ and so $|\sqrt{x_n} - 0| = \sqrt{x_n} \leq \epsilon$. Thus $\sqrt{x_n} \to 0$.

Consider the case that $a > 0$. Suppose that $x_n \geq 0$ for all $k$ and that $x_n \to a > 0$. Note that, for all $n$, we have

$$\left|\sqrt{x_n} - \sqrt{a}\,\right| = \left|\frac{x_n - a}{\sqrt{x_n} + \sqrt{a}}\right| = \frac{|x_n - a|}{\sqrt{x_n} + \sqrt{a}}.$$

Since $x_n \to a$ we can choose $m_1 \in \mathbb{Z}$ so that $n \geq m_1 \Longrightarrow |x_n - a| \leq \frac{3a}{4}$. Then for $n \geq m_1$ we have $\frac{a}{4} \leq x_n \leq \frac{7a}{4}$ so that $\frac{\sqrt{a}}{2} \leq \sqrt{x_n}$. Again since $x_n \to a$, we can choose $m_2 \in \mathbb{Z}$ so that $n \geq m_2 \Longrightarrow |x_n - a| \leq \frac{3\sqrt{a}}{2}\,\epsilon$. Let $m = \max\{m_1, m_2\}$. Then for $n \geq m$ we have $\sqrt{x_n} \geq \frac{\sqrt{a}}{2}$ and we have $|x_n - a| \leq \frac{3\sqrt{a}}{2}\,\epsilon$ and so

$$\left|\sqrt{x_n} - \sqrt{a}\,\right| = \frac{|x_n - a|}{\sqrt{x_n} + \sqrt{a}} \leq \frac{\frac{3\sqrt{a}}{2}\,\epsilon}{\frac{\sqrt{a}}{2} + \sqrt{a}} = \epsilon.$$

Thus $\sqrt{x_n} \to \sqrt{a}$.

**2:** (a) Show that there exist (at least) 3 distinct values of $x$ such that $8x^3 = 6x + 1$.

Solution: Let $f(x) = 8x^3 - 6x - 1$. Notice that $f(x)$ is continuous and we have $f(x) = 0 \leftrightarrow 8x^3 = 6x + 1$. By the Intermediate Value Theorem, since $f(-1) = -3 < 0$ and $f\big(-\frac{1}{2}\big) = 1 > 0$, there is a number $x_1 \in \big(-1, -\frac{1}{2}\big)$ such that $f(x_1) = 0$. Similarly, since $f\big(-\frac{1}{2}\big) = 1 > 0$ and $f(0) = -1 < 0$, there is a number $x_2 \in \big(-\frac{1}{2}, 0\big)$ with $f(x_2) = 0$, and since $f(0) = -1 < 0$ and $f(1) = 1 > 0$, there is a number $x_3 \in (0,1)$ with $f(x_3) = 0$. $\big($In fact, the exact values of $x_1$, $x_2$ and $x_3$ are $x_1 = -\cos(40°)$, $x_2 = -\sin(10°)$ and $x_3 = \cos(20°)\big)$.

(b) Let $f : [0,2] \to \mathbb{R}$ be continuous with $f(0) = f(2)$. Show that $f(x) = f(x+1)$ for some $x \in [0,1]$.

Solution: Let $g(x) = f(x+1) - f(x)$. Note that $g$ is continuous and

$$g(1) = f(2) - f(1) = f(0) - f(1) = -\big(f(1) - f(0)\big) = -g(0).$$

By the Intermediate Value Theorem, there is a number $x \in [0,1]$ with $g(x) = 0$ $\big($indeed if $g(0) \neq 0$ then one of the numbers $g(0)$ and $g(1)$ is positive and the other is negative so there is a number $x \in (0,1)$ with $g(x) = 0\big)$. Then we have $0 = g(x) = f(x+1) - f(x)$ and so $f(x) = f(x+1)$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Show that $f$ is bijective (that is, $f$ is injective and surjective).

Solution: First we note that $f$ is injective since when $x_1 \neq x_2$ we have $|f(x_1) - f(x_2)| \geq |x_1 - x_2| > 0$ so that $f(x_1) \neq f(x_2)$. It remains to show that $f$ is surjective.

We claim that for all $r > 0$, either $\big(\, f(r) \geq f(0) + r$ and $f(-r) \leq f(0) - r\big)$ or $\big(\, f(r) \leq f(0) - r$ and $f(-r) \geq f(0) + r\big)$. Let $r > 0$. Since $|f(r) - f(0)| \geq |r - 0| = r$, it follows that either $f(r) \geq f(0) + r$ or $f(r) \leq f(0) - r$. Likewise, since $|f(-r) - f(0)| \geq |-r - 0| = r$, it follows that either $f(-r) \geq f(0) + r$ or $f(-r) \leq f(0) - r$. Note that in the case that $f(r) \geq f(0) + r$, we must have $f(-r) \leq f(0) - r$ because if we had $f(-r) \geq f(0) + r$ then, by the IVT (applied twice) we could choose $x_1 \in (-r, 0)$ with $f(x_1) = f(0) + \frac{r}{2}$ and we could choose $x_2 \in (0, r)$ with $f(x_2) = f(0) + \frac{r}{2}$ which would give $x_1 \neq x_2$ with $f(x_1) = f(x_2)$ contradicting the fact that $f$ is injective. Similarly, in the case that $f(r) \leq f(0) - r$ we must have $f(-r) \geq f(0) + r$ since if we had $f(-r) \leq f(0) - r$ we could use the IVT to choose $x_1 \in (-r, 0)$ and $x_2 \in (0, r)$ such that $f(x_1) = f(x_2) = f(0) - \frac{r}{2}$. This proves the claim.

Finally, we use the above claim to prove surjectivity. Let $y \in \mathbb{R}$. Choose $r > 0$ such that $f(0) + r > y$ and $f(0) - r < y$. By the claim, either we have $f(-r) \leq f(0) - r < y < f(0) + r \leq f(r)$ or we have $f(r) \leq f(0) - r < y < f(0) + r \leq f(-r)$, and in either case, by the IVT, we can choose $x \in (-r, r)$ such that $f(x) = y$. Thus $f$ is surjective.

**3:** (The Natural Base $e$) Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ for $n \geq 0$ and let $a_n = \left(1 + \frac{1}{n}\right)^n$ for all $n \geq 1$.

(a) Show that $(s_n)_{n\geq 0}$ is increasing and bounded above by 3, and let $e_1 = \lim_{n\to\infty} s_n$.

Solution: Since $s_{n-1} - s_n = \frac{1}{n!} > 0$, it follows that $\langle s_n \rangle$ is strictly increasing. For $n \geq 3$ we have

$$\begin{aligned} s_n &= \sum_{k=0}^{n} \tfrac{1}{k!} = 1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \cdots + \tfrac{1}{n!} \\ &= 2 + \tfrac{1}{2}\left(1 + \tfrac{1}{3} + \tfrac{1}{3\cdot 4} + \tfrac{1}{3\cdot 4\cdot 5} + \cdots + \tfrac{1}{3\cdot 4\cdot\cdots\cdot n}\right) \\ &\leq 2 + \tfrac{1}{2}\left(1 + \tfrac{1}{3} + \tfrac{1}{3^2} + \tfrac{1}{3^3} + \cdots + \tfrac{1}{3^{n-2}}\right) \\ &= 2 + \tfrac{1}{2} \cdot \frac{1 - \frac{1}{3^{n-1}}}{1 - \frac{1}{3}} = 2 + \tfrac{3}{4}\left(1 - \tfrac{1}{3^{n-1}}\right) \leq 2 + \tfrac{3}{4} \leq 3. \end{aligned}$$

Since $(s_n)_{n\geq 0}$ is increasing with $s_n \leq 3$ for all $n$, it converges by the Monotone Convergence Theorem, and by the Comparison Theorem we have $e_1 = \lim_{n\to\infty} s_n \leq 3$.

(b) Use the Binomial Theorem to show that

$$a_n = 1 + 1 + \tfrac{1}{2!}\left(1 - \tfrac{1}{n}\right) + \tfrac{1}{3!}\left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right) + \cdots + \tfrac{1}{n!}\left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right)\cdots\left(1 - \tfrac{n-1}{n}\right).$$

Solution: By the Binomial Theorem, we have

$$\begin{aligned} a_n &= \left(1 + \tfrac{1}{n}\right)^n = 1 + \tbinom{n}{1}\cdot\tfrac{1}{n} + \tbinom{n}{2}\cdot\tfrac{1}{n^2} + \tbinom{n}{3}\cdot\tfrac{1}{n^3} + \cdots + \tbinom{n}{n}\cdot\tfrac{1}{n^n} \\ &= 1 + 1 + \tfrac{n(n-1)}{2!\,n^2} + \tfrac{n(n-1)(n-2)}{3!\,n^3} + \cdots + \tfrac{n(n-1)(n-2)\cdots(1)}{n!\,n^n} \\ &= 1 + 1 + \tfrac{1}{2!}\left(1 - \tfrac{1}{n}\right) + \tfrac{1}{3!}\left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right) + \cdots + \tfrac{1}{n!}\left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right)\cdots\left(1 - \tfrac{n-1}{n}\right). \end{aligned}$$

and we remark that the final term is equal to $\frac{1}{n^n}$.

(c) Show that $(a_n)_{n\geq 1}$ is increasing with $a_n \leq s_n$ for all $n \geq 1$, and let $e_2 = \lim_{n\to\infty} a_n$.

Solution: Using the formula in Part (b) we have

$$\begin{aligned} a_{n+1} &= 1 + 1 + \tfrac{1}{2!}\left(1 - \tfrac{1}{n+1}\right) + \tfrac{1}{3!}\left(1 - \tfrac{1}{n+1}\right)\left(1 - \tfrac{2}{n+1}\right) + \cdots + \tfrac{1}{n!}\left(1 - \tfrac{1}{n+1}\right)\cdots\left(1 - \tfrac{n-1}{n+1}\right) + \tfrac{1}{(n+1)^{n+1}} \\ &\geq 1 + 1 + \tfrac{1}{2!}\left(1 - \tfrac{1}{n}\right) + \tfrac{1}{3!}\left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right) + \cdots + \tfrac{1}{n!}\left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right)\cdots\left(1 - \tfrac{n-1}{n}\right) + \tfrac{1}{(n+1)^{n+1}} \\ &= a_n + \frac{1}{(n+1)^{n+1}} > a_n \end{aligned}$$

for all $n$ and so $(a_n)_{n\geq 1}$ is strictly increasing. Using the same formula again we have

$$\begin{aligned} a_n &= 1 + 1 + \tfrac{1}{2!}\left(1 - \tfrac{1}{n}\right) + \tfrac{1}{3!}\left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right) + \cdots + \tfrac{1}{n!}\left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right)\cdots\left(1 - \tfrac{n-1}{n}\right) \\ &\leq 1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \cdots + \tfrac{1}{n!} = s_n. \end{aligned}$$

Since $(a_n)_{n\geq 1}$ is increasing with $a_n \leq s_n \leq 3$ for all $n$, it converges by the Monotone Convergence Theorem, and by the Comparison Theorem we have $e_2 = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} s_n = e_1$.

(d) Show that $e_2 \geq s_n$ for all $n \geq 0$ and hence $e_2 = e_1$.

Solution: For $k \geq n$ with $n$ fixed, we have (writing the final term in $a_k$ as $\frac{1}{k^k}$)

$$\begin{aligned} a_k &= 1 + 1 + \tfrac{1}{2!}\left(1 - \tfrac{1}{k}\right) + \tfrac{1}{3!}\left(1 - \tfrac{1}{k}\right)\left(1 - \tfrac{2}{k}\right) + \cdots + \tfrac{1}{n!}\left(1 - \tfrac{1}{k}\right)\left(1 - \tfrac{2}{k}\right)\cdots\left(1 - \tfrac{n-1}{k}\right) + \cdots + \tfrac{1}{k^k} \\ &\geq 1 + 1 + \tfrac{1}{2!}\left(1 - \tfrac{1}{k}\right) + \tfrac{1}{3!}\left(1 - \tfrac{1}{k}\right)\left(1 - \tfrac{2}{k}\right) + \cdots + \tfrac{1}{n!}\left(1 - \tfrac{1}{k}\right)\left(1 - \tfrac{2}{k}\right)\cdots\left(1 - \tfrac{n-1}{k}\right) \\ &\longrightarrow 1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \cdots + \tfrac{1}{n!} = s_n \quad \text{as } k \to \infty. \end{aligned}$$

It follows from the Comparison Theorem that $e_2 = \lim_{k\to\infty} a_k \geq s_n$ for all $n$, and hence, by another application of the Comparison Theorem, we have $e_2 \geq \lim_{n\to\infty} s_n = e_1$.

**4:** (a) Show that every sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}$ has a monotonic subsequence (that is either $(x_n)_{n\geq 1}$ has an increasing subsequence or $(x_n)_{n\geq 1}$ has a decreasing subsequence). Hint: consider indices $n$ such that $a_n > a_k$ for all $k > n$.

Solution: For an index $n \geq 1$, let us say that $n$ is a **peak** index of $(x_n)_{n\geq 1}$ when it has the property that $x_n > x_k$ for all $k > n$. Either $(x_n)$ has infinitely many peak indices, or it does not. If $(x_n)$ has infinitely many peak indices, then we can choose peak indices $n_1 < n_2 < n_3 < \cdots$ and then, by the definition of a peak index, $x_{n_1} > x_{n_2} > x_{n_3} > \cdots$. Suppose that $(x_n)$ has only finitely many peak indices. Choose an index $n_1$ which is greater than every peak index. Since $n_1$ is not a peak index, we can choose $n_2 > n_1$ so that $x_{n_2} \geq x_{n_1}$. Since $n_2$ is greater than $n_1$ which is greater than every peak index, $n_2$ is not a peak index and so we can choose $n_3 > n_2$ so that $x_{n_3} \geq x_{n_2}$. We continue this process to obtain indices $n_1 < n_2 < n_3 < \cdots$ with $x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \cdots$.

Alternatively, one can ignore the hint and prove 4(a) using the Bolzano-Weierstrass Theorem. To do this, consider several cases. When $(x_n)$ is not bounded above, construct an increasing subsequence of $(x_k)$. When $(x_n)$ is not bounded below, construct a decreasing subsequence. When $(x_n)$ is bounded, invoke the Bolzano-Weierstrass Theorem to choose a convergent subsequence $(x_{n_k})$ and say $u_k = x_{n_k} \to b$. Then consider the following three cases. Either there exist infinitely many indices $k$ with $u_k = b$ (in this case, construct a constant subsequence of $(u_k)$) or there exist infinitely many indices $k$ with $u_k > b$ (in this case, construct a decreasing subsequence of $(u_k)$) or there exist infinitely many indices $k$ with $u_k < b$ (in this case, construct an increasing subsequence of $(u_k)$).

We also remark that the fact that every sequence in $\mathbb{R}$ has a monotonic subsequence, together with the Monotone Convergence Theorem, immediately imply the Bolzano-Weierstrass Theorem as a corollary. Thus the first solution to this problem supplies you with an alternate (and perhaps easier) proof of the Bolzano-Weierstrass Theorem than the proof we gave (which used the Nested Interval Property of $\mathbb{R}$).

(b) Let $x_n = \frac{n}{\sqrt{2}} - \lfloor \frac{n}{\sqrt{2}} \rfloor$ for $n \geq 1$. Show that $(x_n)_{n\geq 1}$ has a decreasing subsequence $(x_{n_k})_{k\geq 1}$ with $\lim\limits_{k\to\infty} x_{n_k} = 0$. Hint: consider $(1+\sqrt{2})^k$ and $(1-\sqrt{2})^k$.

Solution: By the Binomial Theorem, we have

$$(1+\sqrt{2})^k = 1 + \binom{k}{1}(\sqrt{2}) + \binom{k}{2}(\sqrt{2})^2 + \binom{k}{3}(\sqrt{2})^3 + \binom{k}{4}(\sqrt{2})^4 + \cdots \quad \text{and}$$
$$(1-\sqrt{2})^k = 1 - \binom{k}{1}(\sqrt{2}) + \binom{k}{2}(\sqrt{2})^2 - \binom{k}{3}(\sqrt{2})^3 + \binom{k}{4}(\sqrt{2})^4 - \cdots$$

hence

$$(1+\sqrt{2})^k + (1-\sqrt{2})^k = 2\Big(1 + \binom{k}{2}\cdot 2 + \binom{k}{4}\cdot 2^2 + \binom{k}{6}\cdot 2^3 + \cdots\Big) \quad \text{and}$$
$$(1+\sqrt{2})^k - (1-\sqrt{2})^k = 2\sqrt{2}\Big(\binom{k}{1} + \binom{k}{3}(2) + \binom{k}{5}(2)^2 + \binom{k}{7}2^3 + \cdots\Big),$$

and so we see that $\frac{1}{2}\big((1+\sqrt{2})^k + (1-\sqrt{2})^k\big) \in \mathbb{Z}$ and $\frac{1}{\sqrt{2}}\big((1+\sqrt{2})^k - (1-\sqrt{2})^k\big) \in \mathbb{Z}$ for all $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, let

$$n_k = \tfrac{1}{\sqrt{2}}\big((1+\sqrt{2})^k - (1-\sqrt{2})^k\big)$$

and note that $n_k \in \mathbb{Z}$. Consider the case that $k \in \mathbb{N}$ is odd. Since $-1 < (1-\sqrt{2})^k < 0$ and

$$\tfrac{n_k}{\sqrt{2}} + (1-\sqrt{2})^k = \tfrac{1}{2}\big((1+\sqrt{2})^k - (1-\sqrt{2})^k\big) + (1-\sqrt{2})^k = \tfrac{1}{2}\big((1+\sqrt{2})^k + (1-\sqrt{2})^k\big) \in \mathbb{Z},$$

it follows that $\lfloor \frac{n_k}{\sqrt{2}} \rfloor = \frac{1}{2}\big((1+\sqrt{2})^n + (1-\sqrt{2})^n\big)$. Thus, when $k$ is odd, we have

$$x_{n_k} = \tfrac{n_k}{\sqrt{2}} - \lfloor \tfrac{n_k}{\sqrt{2}} \rfloor = \tfrac{1}{2}\big((1+\sqrt{2})^k - (1-\sqrt{2})^k\big) - \tfrac{1}{2}\big((1+\sqrt{2})^k + (1-\sqrt{2})^k\big) = -(1-\sqrt{2})^k = (\sqrt{2}-1)^k.$$

Thus the subsequence $x_{n_1}, x_{n_3}, x_{n_5}$ of $(x_n)$ is equal to the sequence $(\sqrt{2}-1), (\sqrt{2}-1)^3, (\sqrt{2}-1)^5, \cdots$ which is decreasing with limit 0.

We remark that this is not the only such subsequence. For example, we could have chosen to let $n_k = (1+\sqrt{2})^k + (1-\sqrt{2})^k$ in which case, when $k$ is even we would obtain $\frac{n_k}{\sqrt{2}} - \lfloor \frac{n_k}{\sqrt{2}} \rfloor = \sqrt{2}(\sqrt{2}-1)^k$ giving the subsequence $x_{n_2}, x_{n_4}, x_{n_6}, \cdots$ which would be equal to $\sqrt{2}(\sqrt{2}-1)^2, \sqrt{2}(\sqrt{2}-1)^4, \sqrt{2}(\sqrt{2}-1)^6, \cdots$.

Alternatively, one can ignore the hint and prove the follow more general result. Define $f : \mathbb{R} \to [0, 1)$ by $f(x) = x - \lfloor x \rfloor$ ($f(x)$ is called the **fractional part** of $x$). Let $\alpha \in \mathbb{R}$. Define $x_k = f(\alpha k)$ for $k \geq 0$. If $\alpha \in \mathbb{Q}$ then the sequence $\langle x_k \rangle$ is periodic. If $\alpha \notin \mathbb{Q}$ then

$$\forall a \in [0, 1] \;\; \forall \epsilon > 0 \;\; \forall m \in \mathbb{Z}^+ \;\; \exists k \geq m \;\; |x_k - a| \leq \epsilon \, .$$

We sketch a proof below. We leave it as an exercise to show that 4(b) follows as a corollary.

From the definition of the floor function and the fractional part function $f(x)$, verify that

$$f(x+y) = \begin{cases} f(x) + f(y) & \text{if } f(x) + f(y) < 1 \\ f(x) + f(y) - 1 & \text{if } f(x) + f(y) \geq 1 \end{cases}$$

and

$$f(x-y) = \begin{cases} f(x) - f(y) & \text{if } f(x) \geq f(y) \\ f(x) - f(y) + 1 & \text{if } f(x) < f(y). \end{cases}$$

Since $x_k = f(\alpha k)$, these formulas imply that

$$x_{k_1+k_2} = \begin{cases} x_{k_1} + x_{k_2} & \text{if } x_{k_1} + x_{k_2} < 1 \\ x_{k_1} + x_{k_2} - 1 & \text{if } x_{k_1} + x_{k_2} \geq 1. \end{cases}$$

and

$$x_{k_1-k_2} = \begin{cases} x_{k_2} - x_{k_1} & \text{if } x_{k_2} \geq x_{k_1} \\ x_{k_2} - x_{k_1} + 1 & \text{if } x_{k_2} < x_{k_1}. \end{cases}$$

We wish to prove that when $\alpha \notin \mathbb{Q}$,

$$\forall a \in [0, 1] \;\; \forall \epsilon > 0 \;\; \forall m \in \mathbb{Z}^+ \;\; \exists k \geq m \;\; |x_k - a| \leq \epsilon.$$

Let $a \in [0, 1]$ and let $\epsilon > 0$. Choose $n \in \mathbb{Z}^+$ so that $\frac{1}{n} \geq \epsilon$, then divide the interval $[0, 1]$ into the $n$ subintervals $I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right]$, and note that each of these intervals is of size $\frac{j}{n} - \frac{j-1}{n} = \frac{1}{n}$. Since $a \in [0, 1] = \bigcup_{j=1}^{n} I_j$, we can choose an index $j \in \{1, 2, \cdots, n\}$ such that $a \in I_j$. Since the interval $I_j$ is of size $\frac{1}{n} \leq \epsilon$, it suffices to show that for all $m \in \mathbb{Z}^+$ we can find $k \geq m$ so that $x_k \in I_j$ (because when $x_k$ and $a$ both lie in the same interval $I_j$ we must have $|x_k - a| \leq \frac{1}{n} \leq \epsilon$). It remains for us to show that

$$\forall m \in \mathbb{Z}^+ \;\; \exists k \geq m \;\; x_k \in I_j = \left[\tfrac{j-1}{n}, \tfrac{j}{n}\right].$$

Let $m \in \mathbb{Z}^+$. Choose an index $j_0 \in \{1, 2, \cdots, n\}$ so that for infinitely many indices $k$ we have $x_k \in I_{j_0}$. Choose two indices $k_1, k_2 \in \mathbb{Z}^+$ with $k_2 \geq k_1 + m$ such that $x_{k_1}, x_{k_2} \in I_{j_0}$, and let $l = k_2 - k_1 \geq m$. From our formula for $x_{k_1-k_2}$, we have

$$x_l = x_{k_1-k_2} = \begin{cases} x_{k_2} - x_{k_1} \in \left[0, \frac{1}{n}\right] & \text{if } x_{k_2} \geq x_{k_1} \\ x_{k_2} - x_{k_1} + 1 \in \left[1 - \frac{1}{n}, 1\right] & \text{if } x_{k_2} < x_{k_1}. \end{cases}$$

We have found an index $l \geq m$ such that $x_l \in \left[0, \frac{1}{n}\right] \cup \left[1 - \frac{1}{n}, 1\right]$. We shall show that there is a multiple $k = tl$, where $t \in \mathbb{Z}^+$, such that $x_k \in I_j$ where $I_j$ was the interval that we chose earlier with $a \in I_j$. Since $\alpha \notin \mathbb{Q}$, we have $k\alpha \notin \mathbb{Q}$ for all $k \in \mathbb{Z}^+$ and hence $x_k = f(\alpha k) = \alpha k - \lfloor \alpha k \rfloor \notin \mathbb{Q}$. It follows that $x_l \in \left(0, \frac{1}{n}\right) \cup \left(1 - \frac{1}{n}, 1\right)$. Suppose first that $x_l \in \left(0, \frac{1}{n}\right)$. From our formula for $x_{k_1+k_2}$ we see that $x_{tl} = t\,x_l$ as long as $t\,x_l < 1$. Since $0 < x_k < \frac{1}{n}$, we can choose $t \in \mathbb{Z}^+$ so that $t\,x_l \in I_j$ (to be explicit, verify that if we choose $t = \left\lfloor \frac{j}{n\,x_l} \right\rfloor$ then we have $t\,x_l \in I_j$ ). Then we let $k = tl$ and we have found an index $k \geq m$ such that $x_k \in I_j$. The case that $x_l \in \left(1 - \frac{1}{n}, 1\right)$ is quite similar. If we write $x_l = 1 - \delta$ then we have $0 < \delta < \frac{1}{n}$. From the formula for $x_{k_1+k_2}$ we see that $x_{tl} = 1 - t\delta$ as long as $t\delta \leq 1$. Since $0 < \delta < \frac{1}{n}$, we can choose $t \in \mathbb{Z}^+$ so that $1 - t\delta \in I_j$. Then we let $k = tl$ so that $x_k \in I_j$. This completes the proof that for all $m \in \mathbb{Z}^+$ there exists $k \geq m$ such that $x_k \in I_j$, and the proof of our original claim that

$$\forall a \in [0, 1] \;\; \forall \epsilon > 0 \;\; \forall m \in \mathbb{Z}^+ \;\; \exists k \geq m \;\; |x_k - a| \leq \epsilon \, .$$