PMATH 333 Real Analysis, Solutions to Assignment 3

1: Let $f:\mathbb{R}\to\mathbb{R}$. Suppose that $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbb{R}$.

(a) Show that $f(rx)=rf(x)$ for all $r\in\mathbb{Q}$.

Solution: First we note that $f(0)=f(0+0)=f(0)+f(0)$ and hence (by subtracting $f(0)$ from both sides) we have $f(0)=0$. Next note that for all $x\in\mathbb{R}$ we have $f(x)+f(-x)=f(x+(-x))=f(0)=0$ so that (by subtracting $f(x)$ from both sides) we have $f(-x)=-f(x)$. Let $x\in\mathbb{R}$. We know that $f(0)=0$. Let $k\geq 0$ and suppose, inductively, that $f(kx)=kf(x)$. Then $f((k+1)x)=f(kx+x)=f(kx)+f(x)=kf(x)+f(x)=(k+1)f(x)$. By induction, it follows that $f(kx)=kf(x)$ for all $k\geq 0$. When $k\in\mathbb{Z}$ with $k<0$, letting $\ell=-k>0$ we have $f(\ell x)=\ell f(x)$, hence $f(-\ell x)=-f(\ell x)=-\ell f(x)$, so that $f(kx)=kf(x)$. This proves that $f(kx)=kf(x)$ for all $k\in\mathbb{Z}$. Finally, note that given $x\in\mathbb{R}$ and $r\in\mathbb{Q}$, say $r=\frac{k}{n}$ with $k\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$, we have $nf\big(\frac{k}{n}x\big)=f\big(n\cdot\frac{k}{n}\,x\big)=f(kx)=k\,f(x)$ so that (by dividing by n) we have $f\big(\frac{k}{n}\,x\big)=\frac{k}{n}\,f(x)$, that is $f(rx)=rf(x)$.

(b) Show that f is continuous at 0 if and only if f is continuous on $\mathbb{R}$.

Solution: If f is continuous on $\mathbb{R}$ then of course f is continuous at 0. Suppose that f is continuous at 0. We claim that f is uniformly continuous on $\mathbb{R}$. Let $\epsilon>0$. Since f is continuous at 0 with $f(0)=0$, we can choose $\delta>0$ such that for all $t\in\mathbb{R}$, if $|t|=|t-0|<\delta$ then $|f(t)|=|f(t)-f(0)|<\epsilon$. Let $x,y\in\mathbb{R}$ with $|x-y|<\delta$. By the choice of δ (taking $t=x-y$), we have $|f(y-x)|<\epsilon$. Using part (a), we have $\big|f(x)-f(y)\big|=\big|f(x)+f(-y)\big|=\big|f(x+(-y))\big|=\big|f(x-y)\big|<\epsilon$. Thus f is uniformly continuous, as claimed.

(c) Show that if f is continuous at 0 then there exists $m\in\mathbb{R}$ such that $f(x)=mx$ for all $x\in\mathbb{R}$.

Solution: Suppose f is continuous at 0. By part (b), f is continuous (indeed uniformly continuous) on $\mathbb{R}$. Let $m=f(1)$. By part (a), for every $r\in\mathbb{Q}$ we have $f(r)=f(r\cdot 1)=rf(1)=mr$. Given $x\in\mathbb{R}$, we can choose a sequence $(t_n)_{n\geq 1}$ in $\mathbb{Q}$ such that $t_n\to x$ in $\mathbb{R}$. Since $t_n\in\mathbb{Q}$, we have $f(t_n)=mt_n$, and since $t_n\to x$ and f is continuous at x we have $f(x)=\lim\limits_{n\to\infty}f(t_n)=\lim\limits_{n\to\infty}mt_n=m\lim\limits_{n\to\infty}t_n=mx$, as required.

2: (a) Define $f,g:\mathbb{R}\to\mathbb{R}$ by $f(x)=x^3$ and $g(x)=\sqrt[3]{x}$. Show that g is uniformly continuous but f is not.

Solution: We claim that $f(x)$ is not uniformly continuous. Choose $\epsilon=1$. Let $\delta>0$ Choose $a=\frac{1}{\delta}$ and $x=\delta+\frac{1}{\delta}$. Then $|x-a|=\delta$ and we have

$$\big|f(x)-f(a)\big|=\left(\delta+\tfrac{1}{\delta}\right)^3-\left(\tfrac{1}{\delta}\right)^3=3\delta+3\cdot\tfrac{1}{\delta}+\delta^3>3\left(\delta+\tfrac{1}{\delta}\right)>3>\epsilon$$

because when $\delta\ge 1$ we have $\delta+\frac{1}{\delta}>\delta\ge 1$ and when $0<\delta\le 1$ we have $\delta+\frac{1}{\delta}>\frac{1}{\delta}\ge 1$. Thus f is not uniformly continuous.

We claim that g is uniformly continuous. We provide two proofs. For the first proof, we use only the definition of uniform continuity (and some algebra). First we note that for $\delta>0$ and for $a,x\in\mathbb{R}$, in the case that $|a|\le 2\delta$, when $|x-a|<\delta$ we have $|x|<3\delta$ and so

$$|f(x)-f(a)|\le|f(x)|+|f(a)|<(2\delta)^{1/3}+(3\delta)^{1/3}=(2^{1/3}+3^{1/3})\,\delta^{1/3}<3\,\delta^{1/3}$$

(because $3<\frac{27}{8}=\left(\frac{3}{2}\right)^3$ so that $3^{1/3}<\frac{3}{2}$ and hence $2^{1/3}+3^{1/3}<2\cdot\frac{3}{2}=3$) and in the case that $|a|\ge 2\delta$, when $|x-a|<\delta$, the numbers a and x have the same sign and we have $|x|\ge\delta$ and so

$$\begin{aligned}|f(x)-f(a)|=|x^{1/3}-a^{1/3}|&=\left|\frac{x-a}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}}\right|=\frac{|x-a|}{|x|^{2/3}+|x|^{1/3}|a|^{1/3}+|a|^{2/3}}\\&<\frac{\delta}{\delta^{2/3}+\delta^{1/3}(2\delta)^{1/3}+(2\delta)^{2/3}}=\frac{\delta^{1/3}}{1+2^{1/3}+4^{1/3}}<\delta^{1/3}<3\,\delta^{1/3}.\end{aligned}$$

Thus given $\epsilon>0$ we can choose $\delta=\frac{1}{27}\,\epsilon^3$ so that $3\,\delta^{1/3}=\epsilon$ and then for all $a,x\in\mathbb{R}$ with $|x-a|<\delta$ we have $|f(x)-f(a)|<3\,\delta^{1/3}=\epsilon$. Thus g is uniformly continuous.

For the second proof, we shall use the fact that a function which is continuous on a closed bounded inteval is uniformly continuous. Let $\epsilon>0$. Since the restriction of f to $[0,2]$ is continuous, it is uniformly continuous, so we can choose $\delta_1>0$ such that for all $x,y\in[0,2]$, if $|x-y|<\delta_1$ then $|f(x)-f(y)|<\epsilon$. For all $x\in[1,\infty)$, we have $f'(x)=\frac{1}{3}^{-2/3}$, which is positive and decreasing, and hence $|f'(x)|=f'(x)\le f'(1)=\frac{1}{3}$. Let $\delta_2=3\epsilon$ and note that for all $x,y\ge 1$ with $|x-y|<\delta_2$, by the MVT we can choose t between x and y such that $f(x)-f(y)=f'(t)(x-y)$, and then $|f(x)-f(y)|\le|f'(t)||x-y|\le\frac{1}{3}|x-y|<\frac{1}{3}\delta_2=\epsilon$. Finally, we choose $\delta=\min(1,\delta_1,\delta_2)$. Let $x,y\in\mathbb{R}$ with $|x-y|<\delta$. Since $|x-y|<1$ (so that we cannot have $x<1$ and $y>2$ or vice versa) it follows that either $x,y\in[0,2]$ or $x,y\in[1,\infty)$, and in either case we have $|f(x)-f(y)|<\epsilon$, as required.

(b) Let $f:(a,b)\subseteq\mathbb{R}\to\mathbb{R}$. Show that f is uniformly continuous on (a,b) if and only if there exists a continuous function $g:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ such that $g(x)=f(x)$ for all $x\in(a,b)$.

Solution: If $g:[a,b]\to\mathbb{R}$ is continuous, hence uniformly continuous, and $f(x)=g(x)$ for all $x\in(a,b)$, then of course f is uniformly continuous on (a,b) (given $\epsilon>0$ we choose $\delta>0$ such that $|g(x)-g(y)|<\epsilon$ for all $x,y\in[a,b]$, then of course we have $|f(x)-f(y)|=|g(x)-g(y)|<\epsilon$ for all $x,y\in(a,b)$).

Suppose, conversely, that f is uniformly continuous on (a,b). Choose a sequence $(x_n)_{n\ge1}$ in (a,b) with $x_n\to a$. We claim that $(f(x_n))_{n\ge1}$ is Cauchy. Let $\epsilon>0$. Since f is uniformly continuous we can choose $\delta>0$ so that $|x-y|<\delta\implies|f(x)-f(y)|<\epsilon$. Since $(x_n)_{n\ge1}$ converges, it is Cauchy, so we can choose $N\in\mathbb{Z}^+$ such that $n,n\ge N\implies|x_n-x_m|<\delta$. Then when $n,m\ge N$ we have $|x_n-x_m|<\delta$ so that $|f(x_n)-f(x_m)|<\epsilon$. Thus $(f(x_n))_{n\ge1}$ is Cauchy, as claimed. Since $(f(x_n))_{n\ge1}$ is Cauchy, it converges, say $c=\lim\limits_{n\to\infty}f(x_n)$. Choose a sequence $(y_n)_{n\ge1}$ in (a,b) with $y_n\to b$. The same argument used above shows that $(f(y_n))_{n\ge1}$ is Cauchy, so it converges, and we let $d=\lim\limits_{n\to\infty}f(y_n)$.

Define $g:[a,b]\to\mathbb{R}$. by $g(x)=f(x)$ for all $x\in(a,b)$ and by $g(a)=c$ and $g(b)=d$. We claim that g is continuous at a. Let $\epsilon>0$. By the uniform continuity of f we can choose δ with $0<\delta<b-a$ such that for all $x,y\in(a,b)$ with $|x-y|<\delta$ we have $|f(x)-f(y)|<\frac{\epsilon}{2}$. Since $x_n\to a$ and $f(x_n)\to c$ we can choose n so that $|x_n-a|<\delta$ and $|f(x_n)-c|<\frac{\epsilon}{2}$. Let $x\in[a,b]$ with $|x-a|<\delta$. If $x=a$ then of course $|g(x)-g(a)|=0<\epsilon$. If $x\ne a$ then we have $x\in(a,a+\delta)$ and $x_n\in(a,a+\delta)$ so that $|x-x_n|<\delta$, hence $|f(x)-f(x_n)|<\frac{\epsilon}{2}$. Thus we have $|g(x)-g(a)|=|f(x)-c|\le|f(x)-f(x_n)|+|f(x_n)-c|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$. This proves that g is continuous at a, as claimed. A similar argument shows that g is continuous at b.

3: (a) Let $f, g : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be integrable and let $c \in \mathbb{R}$. Prove, from Definition 3.3, that the functions $f+g$ and cf are integrable on $[a,b]$ with $\int_a^b (f+g) = \int_a^b f + \int_a^b g$ and $\int_a^b cf = c\int_a^b f$.

Solution: Let $I = \int_a^b f$ and $J = \int_a^b g$. We claim that the function $f+g$ is integrable with $\int_a^b (f+g) = I + J$. Let $\epsilon > 0$. Choose $\delta_1 > 0$ such that for all partitions X of $[a,b]$ with $|X| < \delta_1$ we have $|S - I| < \frac{\epsilon}{2}$ for all Riemann sums S for f os X, and choose $\delta_2 > 0$ such that for all partitions X of $[a,b]$ with $|X| < \delta_2$ we have $|T - J| < \epsilon_2$ for all Riemann sums T for g on X. Let $\delta = \min(\delta_1, \delta_2)$. Let $X = \{x_0, x_1, \cdots, x_n\}$ be any partition of $[a,b]$ with $|X| < \delta$. Let $t_k \in [x_{k-1}, x_k]$ be any sample points. Then we have

$$\begin{aligned}\Big|\sum_{k=1}^n (f(t_k)+g(t_k))\Delta_k x - (I+J)\Big| &= \Big|\big(\sum_{k=1}^n f(t_k)\Delta_k x - I\big) + \big(\sum_{k=1}^n g(t_k)\Delta_k x - J\big)\Big| \\ &\le \Big|\sum_{k=1}^n f(t_k)\Delta_k x - I\Big| + \Big|\sum_{k=1}^n g(t_k)\Delta_k x - J\big)\Big| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.\end{aligned}$$

Thus $f+g$ is integrable with $\int_a^b (f+g) = I + J$, as claimed.

We claim that cf is integrable on $[a,b]$ with $\int_a^b cf = cI$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for all partitions X of $[a,b]$ with $|X| < \delta$ we have $|S - I| < \frac{\epsilon}{|c|+1}$ for every Riemann sum S for f on X. Let $X = \{x_0, x_1, \cdots, x_n\}$ be any partition for $[a,b]$ with $|X| < \delta$. Let $t_k \in [x_{k-1}, x_k]$ be any sample points. Then

$$\Big|\sum_{k=1}^n (cf)(t_k)\Delta_k x - cI\Big| = |c|\,\Big|\sum_{k=1}^n f(t_k)\Delta_k x\Big| < |c|\,\tfrac{\epsilon}{|c|+1} < \epsilon.$$

Thus cf is integrable with $\int_a^b cf = cI$, as required (we used $\frac{\epsilon}{|c|+1}$ rather that $\frac{\epsilon}{|c|}$ to include the case $c = 0$).

(b) Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be any function (not necessarily bounded). Show that if f is integrable (according to Definition 3.3, but without the assumption that f is bounded) then f must be bounded.

Solution: Suppose that f is integrable on $[a,b]$ and let $I = \int_a^b f$. Suppose, for a contradiction, that f is not bounded, say f is not bounded above (the case that f is not bounded below is similar). This means that for every $L > 0$ we can choose $u \in [a,b]$ such that $f(u) > L$. Since f is integrable, by taking $\epsilon = 1$ in the definition of integrability, we can choose $\delta > 0$ such that for every partition X of $[a,b]$ with $|X| < \delta$, we have $|S - I| < 1$ for every Riemann sum S for f on X. Let $X = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a,b]$ with $|X| < \delta$, and let $d = \min\big\{\Delta_k x \,\big|\, 1 \le k \le n\big\}$. Choose sample points $s_k \in [x_{k-1}, x_k]$ and let S be the Riemann sum $S = \sum_{k=1}^n f(s_k)\Delta_k x$. Let $M = \max\big\{f(s_1), f(s_2), \cdots, f(s_n)\big\}$ and choose $u \in [a,b]$ such that $f(u) > M + \frac{2}{d}$. Note that u must lie in one of the sub-intervals, so we can choose an index ℓ such that $u \in [x_{\ell-1}, x_\ell]$. Choose new sample points, letting $t_k = s_k$ whenever $k \neq \ell$ and letting $t_\ell = u$, and let T be the resulting Riemann sum $T = \sum_{k=1}^n f(t_k)\Delta_k x$. Since S and T are both Riemann sums for f on X, we have $|S - I| < 1$ and $|T - I| < 1$ so that $|T - S| < 2$. But

$$|T - S| = \sum_{k=1}^n \big(f(t_k) - f(s_k)\big)\Delta_k x = (f(u) - f(s_\ell))\Delta_\ell x > \big((M + \tfrac{2}{d})) - M\big)\Delta_\ell x = \tfrac{2}{d}\cdot\Delta_\ell \ge \tfrac{2}{d}\cdot d = 2.$$

4: Determine (with proof) which of the following statements are true for all bounded functions $f:[0,1]\to\mathbb{R}$.

(a) If $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{n}\,f\big(\frac{k}{n}\big)$ exists then f is integrable on $[0,1]$.

Solution: This is FALSE. For example, let $f:[0,1]\to\mathbb{R}$ be the function from Example 3.4 in the Lecture Notes given by $f(x)=1$ when $x\in\mathbb{Q}$ and $f(x)=0$ when $x\notin\mathbb{Q}$. As shown in Example 3.4, this function is not integrable, but we have $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{n}\,f\big(\frac{k}{n}\big)=\lim\limits_{n\to\infty}\big(\sum\limits_{k=1}^{n}\frac{1}{n}\cdot 1\big)=\lim\limits_{n\to\infty}1=1$.

(b) For all $a,b\in[0,1]$ and $S\in\mathbb{R}$, if $f(a)<S<f(b)$ then there exists a partition $X=\{x_0,x_1,\cdots,x_n\}$ of $[0,1]$ and there exist sample points $t_k\in[x_{k-1},x_k]$ such that $\sum\limits_{k=1}^{n}f(t_k)\Delta_k x=S$.

Solution: This is TRUE. Let $a,b\in[0,1]$ and $S\in\mathbb{R}$ with $f(a)<S<f(b)$.

Suppose first that one of $f(0)$ and $f(1)$ is less than or equal to S and that the other is greater than or equal to S, let us say that $f(0)\le S\le f(1)$. If $f(0)=S$ then S is equal to the Riemann sum for f on the partition $X=\{x_0,x_1\}=\{0,1\}$ with sample point $t_1=0$. Similarly, if $f(1)=S$ then S is equal to the Riemann sum for f on $X=\{x_0,x_1\}=\{0,1\}$ with $t_1=1$. Suppose that $f(0)<S<f(1)$. Let $c=\frac{f(1)-S}{f(1)-f(0)}$ and note that $0<c<1$. Let X be the partition $X=\{x_0,x_1,x_2\}=\{0,c,1\}$ and let $t_1=0$ and $t_2=1$. Then

$$\sum_{k=1}^{2}f(t_k)\Delta_k x=f(0)c+f(1)(1-c)=f(0)\,\tfrac{f(1)-S}{f(1)-f(0)}+f(1)\,\tfrac{S-f(0)}{f(1)-f(0)}=S\,.$$

Now suppose that either both $f(0)$ and $f(1)$ are smaller that S, or they are both larger than S, let us say they are both smaller (the case that they are both larger is similar, but interchanges the roles of a and b). We have $f(0)<S$ and $f(1)<S$ and $S<f(b)$. Note that $b\neq 0,1$ so we have $b\in(0,1)$. Let

$$c=\tfrac{bf(b)-bS)}{f(b)-f(0)}\quad\text{and}\quad d=\tfrac{(1-b)S+bf(b)-f(1)}{f(b)-f(1)}$$

(c and d are chosen so that $cf(0)+(b-c)f(b)=bS$ and $(d-b)f(b)+(1-d)f(1)=(1-b)S$). Since $S>f(0)$ we have $c<\frac{bf(b)-bf(0)}{f(b)-f(0)}=b$ and since $S<f(b)$ we have $c>0$ so that $0<c<b$. Since $S<f(b)$ we have $d<\frac{(1-b)f(b)+bf(b)-f(1)}{f(b)-f(1)}=1$ and since $S>f(1)$ we have $d>\frac{(1-b)f(1)+bf(b)-f(1)}{f(b)-f(1)}=b$ so that $b<d<1$. Let X be the partition $X=\{x_0,x_1,x_2,x_3\}=\{0,c,d,1\}$ and use the sample points $t_1=0$, $t_2=b$ and $t_3=1$. Then

$$\begin{aligned}\sum_{k=1}^{3}f(t_k)\Delta_k x&=f(0)(c-0)+f(b)(d-c)+f(1)(1-d)=(f(b)-(1))d-(f(b)-f(0))c+f(1)\\&=\big((1-b)S+bf(b)-f(1)\big)-\big(bf(b)-bS\big)+f(1)=S.\end{aligned}$$