PMATH 333 Real Analysis, Assignment 4 Due Mon Oct 4, 11:00 pm

1: (a) Find $\displaystyle\int_0^1 e^x\,dx$ by evaluating the limit of a sequence of Riemann sums for the function $f(x) = e^x$.

(b) Find $\displaystyle\int_1^4 \sqrt{x}\,dx$ by evaluating the limit of a sequence of Riemann sums for the function $f(x) = \sqrt{x}$.

2: (a) Let f be integrable on $[a,b]$. Show that g is integrable on $[a,b]$, where $g(x) = \left\{ \begin{array}{cl} f(x) & \text{, if } f(x) \geq 0 \\ 0 & \text{, if } f(x) < 0 \end{array} \right\}$.

(b) Show that f is integrable on $[0,1]$, where f is defined by

$$f(x) = \left\{ \begin{array}{l} \frac{1}{2^\ell} \text{ , if } x = \frac{k}{2^\ell} \text{ for some positive integers } k, \ell \text{ with } k \text{ odd} \\ 0 \text{ , otherwise} \end{array} \right\}.$$

3: Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ and let $f : (a,b) \subseteq \mathbb{R} \to \mathbb{R}$ be continuous. We say that the improper integral $\int_a^b f$ *converges* when for some (hence for any) $c \in (a,b)$, the limits $\displaystyle\lim_{r \to a} \int_r^c f$ and $\displaystyle\lim_{s \to b} \int_c^s f$ both exist and are finite.

(a) Determine whether $\displaystyle\int_1^\infty \frac{\sin\left(\frac{1}{x}\right)}{\sqrt{\ln x}}\,dx$ converges.

(b) Determine whether $\displaystyle\int_2^\infty \ln\left(\sec \tfrac{\pi}{x}\right)\,dx$ converges.

4: (a) Let $f : [a,b] \to [c,d]$ be bijective and decreasing with $f(a) = d$ and $f(b) = c$. Let $g = f^{-1} : [c,d] \to [a,b]$. Suppose f and g are continuous and consider the area of the region $a \leq x \leq b$, $c \leq y \leq f(x)$. Prove that

$$\int_{x=a}^b \big(f(x) - c\big)\,dx = \int_{y=c}^d \big(g(y) - a\big)\,dy$$

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(b) Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$. For a partition $X = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$, define

$$\text{Length}(f, X) = \sum_{k=1}^n \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$$

then define the *length* of the graph $y = f(x)$ on $[a,b]$ to be

$$\text{Length}(f) = \sup\left\{ \text{Length}(f, X) \,\middle|\, X \text{ is a partition for } [a,b] \right\}$$

(the above supremum can be finite or infinite). We say that f is *rectifiable* on $[a,b]$ when Length(f) is finite. Show that if f is rectifiable on $[a,b]$ then f is integrable on $[a,b]$.