PMATH 333 Real Analysis, Solutions to Assignment 4

1: (a) Find $\displaystyle\int_0^1 e^x\,dx$ by evaluating the limit of a sequence of Riemann sums for the function $f(x)=e^x$.

Solution: Let $X_n=\{x_0,x_1,\cdots,x_n\}$ where $x_i=\frac{1}{n}\,i$. Then $\Delta_i x=\frac{1}{n}$ for all i so $||X_n||=\frac{1}{n}$ and so $||X_n||\to 0$ as $n\to\infty$. Let $S_n=\sum\limits_{i=1}^{n} f(t_i)\Delta_i x$ where $t_i=x_i$. Then

$$\begin{aligned}
S_n &= \sum_{i=1}^{n} f(x_i)\Delta_i x \\
&= \sum_{i=1}^{n} f\left(\tfrac{1}{n}\,i\right)\left(\tfrac{1}{n}\right) \\
&= \sum_{i=1}^{n} e^{(1/n)\,i}\left(\tfrac{1}{n}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(e^{1/n}\right)^i \\
&= \frac{1}{n}\,\frac{e^{1/n}\left(\left(e^{1/n}\right)^n-1\right)}{e^{1/n}-1} \\
&= e^{1/n}(e-1)\,\frac{1/n}{e^{1/n}-1}
\end{aligned}$$

By l'Hôpital's Rule we have $\displaystyle\lim_{n\to\infty}\frac{1/n}{e^{1/n}-1}=\lim_{n\to\infty}\frac{-\frac{1}{n^2}}{-\frac{1}{n^2}\,e^{1/n}}=\lim_{n\to\infty}\frac{1}{e^{1/n}}=1$, and so

$$\int_0^1 e^x\,dx=\lim_{n\to\infty}S_n=\lim_{n\to\infty}e^{1/n}(e-1)\,\frac{1/n}{e^{1/n}-1}=e-1\,.$$

(b) Find $\displaystyle\int_1^4 \sqrt{x}\,dx$ by evaluating the limit of a sequence of Riemann sums for the function $f(x)=\sqrt{x}$.

Solution: Let $f(x)=\sqrt{x}$ on $[1,4]$. Note that the range of f is $[1,2]$. For $n\in\mathbb{Z}^+$, let $Y_n=\{y_0,y_1,\cdots,y_n\}$ be the partition of the range $[1,2]$ into n equal sub-intervals, so we have $y_k=1+\frac{k}{n}$, let $X_n=\{x_0,x_1,\cdots,x_n\}$ be the corresponding partition of the domain $[1,4]$ given by $x_k={y_k}^2=1+\frac{2k}{n}+\frac{k^2}{n^2}$, and let $t_k=x_k$. Note that $\Delta_k x=(x_k-x_{k-1})=\left(1+\frac{2k}{n}+\frac{k^2}{n^2}\right)-\left(1+\frac{2k-2}{n}+\frac{k^2-2k+1}{n^2}\right)=\frac{2}{n}+\frac{2k}{n^2}-\frac{1}{n^2}$, which is increasing with k so that $|X_n|=\Delta_n x=\frac{4}{n}-\frac{1}{n^2}\to 0$ as $n\to\infty$. Thus

$$\begin{aligned}
\int_1^4\sqrt{x}\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(t_k)\Delta_k x=\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{x_k}\,\Delta_k x \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\tfrac{k}{n}\right)\left(\tfrac{2}{n}+\tfrac{2k}{n^2}-\tfrac{1}{n^2}\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\tfrac{2}{n}+\tfrac{2k}{n^2}-\tfrac{1}{n^2}+\tfrac{2k}{n^2}+\tfrac{2k^2}{n^3}-\tfrac{k}{n^3}\right) \\
&= \lim_{n\to\infty}\left(\left(\tfrac{2}{n}-\tfrac{1}{n^2}\right)\sum_{k=1}^{n}1+\left(\tfrac{4}{n^2}-\tfrac{1}{n^3}\right)\sum_{k=1}^{n}k+\tfrac{2}{n^3}\sum_{k=1}^{n}k^3\right) \\
&= \lim_{n\to\infty}\left(\tfrac{2n-1}{n^2}\cdot n+\tfrac{4n-1}{n^3}\cdot\tfrac{n(n+1)}{2}+\tfrac{2}{n^3}\cdot\tfrac{n(n+1)(2n+1)}{6}\right) \\
&= 2+2+\tfrac{2}{3}=\tfrac{14}{3}\,.
\end{aligned}$$

2: (a) Let f be integrable on $[a,b]$. Show that g is integrable on $[a,b]$, where $g(x) = \left\{ \begin{array}{ll} f(x) & \text{, if } f(x) \geq 0 \\ 0 & \text{, if } f(x) < 0 \end{array} \right\}$.

Solution: Let $\epsilon > 0$. Since f is integrable, we can choose a partition $X = (x_0, x_1, \cdots, x_n)$ of $[a,b]$ such that $U(f,X) - L(f,X) < \epsilon$. For $1 \leq k \leq n$, let $M_k(g) = \sup\left\{g(t) \,\middle|\, t \in [x_{k-1}, x_k]\right\}$, $m_k(g) = \inf\left\{g(t) \,\middle|\, t \in [x_{k-1}, x_k]\right\}$ and similarly for $M_k(f)$ and $m_k(f)$. We claim that $M_k(g) - m_k(g) \leq M_k(f) - m_k(f)$ for all $1 \leq k \leq n$. Fix k with $1 \leq k \leq n$. If $f(x) \leq 0$ for all $x \in [x_{k-1}, x_k]$ then we have $g(x) = 0$ for all $x \in [x_{k-1}, x_k]$ so that $M_k(g) = m_k(g) = 0$, and hence $M_k(g) - m_k(g) = 0 \leq M_k(f) - m_k(f)$, as claimed. Suppose that $f(x) > 0$ for some $x \in [x_{k-1}, x_k]$. Since $g(x) \geq f(x)$ for all x, we have $m_k(g) \geq m_k(f)$. Since there exists $x \in [x_{k-1}, x_k]$ such that $f(x) > 0$, it follows that

$$M_k(f) = \sup\left\{f(t) \,\middle|\, t \in [x_{k-1}, x_k]\right\} = \sup\left\{f(t) \,\middle|\, t \in [x_{k-1}, x_k], f(t) > 0\right\} = \sup\left\{g(t) \,\middle|\, t \in [x_{k-1}, x_k]\right\} = M_k(g).$$

Since $M_k(g) = M_k(f)$ and $m_k(g) \geq m_k(f)$ we have $M_k(g) - m_k(g) \leq M_k(f) - m_k(f)$, as claimed. Thus

$$U(g,X) - L(g,X) = \sum_{k=1}^{n} \big(M_k(g) - m_k(g)\big)\Delta_k x \leq \sum_{k=1}^{n} \big(M_k(f) - m_k(f)\big)\Delta_k x = U(f,X) - L(f,X) < \epsilon.$$

It follows that g is integrable, as required (by Part 2 of Theorem 1.16).

(b) Show that f is integrable on $[0,1]$, where f is defined by

$$f(x) = \left\{ \begin{array}{l} \frac{1}{2^\ell} \text{ , if } x = \frac{k}{2^\ell} \text{ for some positive integers } k, \ell \text{ with } k \text{ odd} \\ 0 \text{ , otherwise} \end{array} \right\}.$$

Solution: Let $\epsilon > 0$. Choose $\ell > 0$ so that $\frac{1}{2^\ell} < \frac{\epsilon}{2}$. Note that there are exactly $2^\ell - 1$ points $x \in [0,1]$ for which $f(x) \geq \frac{1}{2^\ell}$, namely the points $x = \frac{j}{2^\ell}$ with $1 \leq j < 2^\ell$ (including even values of j). Choose a partition $X = (x_0, x_1, \cdots, x_n)$ with $|X| < \frac{1}{4^\ell}$ such that each of the $2^\ell - 1$ points $\frac{j}{2^\ell}$ lies in the interior (but is not an endpoint) of one of the subintervals $[x_{k-1}, x_k]$. Let $A \subseteq \{1, 2, \cdots, n\}$ be the set of indices k such that $[x_{k-1}, x_k]$ contains one of the points $\frac{j}{2^\ell}$, and let $B = \{1, \cdots, n\} \setminus A$ be the set of all other indices k. For each $k \in A$ we have $M_k \leq 1$ $\big($indeed $M_k \leq \frac{1}{2}\big)$ and $m_k = 0$ so that $(M_k - m_k)\Delta_k x \leq \Delta_k x \leq \frac{1}{4^\ell}$, and hence

$$\sum_{k \in A} (M_k - m_k)\Delta_k x \leq (2^\ell - 1) \cdot \frac{1}{4^\ell} < \frac{1}{2^\ell} < \frac{\epsilon}{2}\,.$$

For each $k \in B$ we have $M_k \leq \frac{1}{2^\ell}$ and $m_k = 0$ so that $(M_k - m_k)\Delta_k x \leq \frac{1}{2^\ell}\Delta_k x$, and hence

$$\sum_{k \in B} (M_k - m_k)\Delta_k x \leq \frac{1}{2^\ell} \sum_{k \in B} \Delta_k x \leq \frac{1}{2^\ell} \sum_{k=1}^{n} \Delta_k x = \frac{1}{2^\ell} < \frac{\epsilon}{2}\,.$$

Thus

$$U(h,X) - L(h,X) = \sum_{k=1}^{n} (M_k - m_k)\Delta_k x = \sum_{k \in A} (M_k - m_k)\Delta_k x + \sum_{k \in B} (M_k - m_k)\Delta_k x < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It follows that h is integrable, as required (by Part 2 of Theorem 1.16).

3: Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ and let $f : (a, b) \subseteq \mathbb{R} \to \mathbb{R}$ be continuous. We say that the improper integral $\int_a^b f$ *converges* when for some (hence for any) $c \in (a, b)$, the limits $\lim_{r \to a} \int_r^c f$ and $\lim_{s \to b} \int_c^s f$ both exist and are finite.

(a) Determine whether $\displaystyle\int_1^\infty \frac{\sin\left(\frac{1}{x}\right)}{\sqrt{\ln x}}\, dx$ converges.

Solution: For $0 \leq \theta \leq \frac{\pi}{6}$, since $\sin\theta$ is concave down with $\sin 0 = 0$ and $\sin\frac{\pi}{6} = \frac{1}{2}$ we have $\sin\theta \geq \frac{3}{\pi\theta}$. When $2 \leq x < \infty$ we have $0 < \frac{1}{x} \leq \frac{1}{2} < \frac{\pi}{6}$ so that $\sin\left(\frac{1}{x}\right) \geq \frac{3}{\pi x}$. Thus

$$\int_2^\infty \frac{\sin(\frac{1}{x})}{\sqrt{\ln x}}\, dx \geq \int_2^\infty \frac{3\, dx}{\pi x\sqrt{\ln x}} = \left[\tfrac{6}{\pi}\sqrt{\ln x}\right]_2^\infty = \infty.$$

Thus the given integral diverges.

The solution is complete, but we remark that for $1 \leq x$ we have $0 < \frac{1}{x} \leq 1 \leq \frac{\pi}{3}$ so that $0 < \sin\left(\frac{1}{x}\right) \leq \frac{\sqrt{3}}{2}$ and, for $1 \leq x \leq 2$, since $\ln x$ is concave down, we have $\ln x \geq \ln 2(x-1)$, and so

$$\int_1^2 \frac{\sin(\frac{1}{x})}{\sqrt{\ln x}}\, dx \leq \int_1^2 \frac{\sqrt{3}\, dx}{2\sqrt{\ln x}} \leq \int_1^2 \frac{\sqrt{3}\, dx}{2\sqrt{\ln 2(x-1)}} = \left[\tfrac{\sqrt{3}}{\sqrt{\ln 2}}\sqrt{x-1}\right]_{1^+}^2 = \frac{\sqrt{3}}{\sqrt{\ln 2}} < \infty.$$

(b) Determine whether $\displaystyle\int_2^\infty \ln\left(\sec\frac{\pi}{x}\right)\,dx$ converges.

Solution: For $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$, since $\cos\theta$ is concave down with $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\cos\frac{\pi}{2} = 0$, it follows that $\cos\theta \ge \frac{3}{\pi}\left(\frac{\pi}{2} - \theta\right) = \frac{3}{2} - \frac{3}{\pi}\theta$. When $2 \le x \le 3$ we have $\frac{\pi}{3} \le \frac{\pi}{x} \le \frac{\pi}{2}$ so that $\cos\frac{\pi}{x} \ge \frac{3}{2} - \frac{3}{x} = \frac{3(x-2)}{2x}$, and hence $-\ln\left(\cos\frac{\pi}{x}\right) \le -\ln\frac{3(x-2)}{2x} = \ln\frac{2x}{3(x-2)}$. Thus, by integrating by parts using $u = \ln\frac{2x}{3(x-2)}$ and $v = x$, noting that $du = \frac{3(x-2)}{2x}\cdot\frac{2}{3}\cdot\frac{(x-2)-x}{(x-2)^2}\,dx = \frac{-2}{x(x-2)}\,dx$, we have

$$\int \ln\left(\sec\frac{\pi}{x}\right)dx \le \int \ln\frac{2x}{3(x-2)}\,dx = x\ln\frac{2x}{3(x-2)} + \int \frac{2}{x-2}\,dx = x\ln\frac{2x}{3(x-2)} + 2\ln(x-2)$$

$$\int_2^3 \ln\left(\sec\frac{\pi}{x}\right)dx = \Big[x\ln\frac{2x}{3} - (x-2)\ln(x-2)\Big]_{2^+}^{3} = 3\ln 2 - 2\ln\frac{4}{3} = 2\ln 3 - \ln 2 = \ln\frac{9}{2},$$

where we used the fact that, by l'Hôpital's Rule, we have

$$\lim_{x\to 2^+}(x-2)\ln\big(3(x-2)\big) = \lim_{u\to 0^+} u\ln 3u = \lim_{u\to 0^+}\frac{\ln 3u}{\frac{1}{u}} = \lim_{u\to 0^+}\frac{\frac{1}{u}}{-\frac{1}{u^2}} = \lim_{u\to 0^+}(-u) = 0.$$

For $g(\theta) = \theta - \sin\theta$ we gave $g(0) = 0$ and $g'(\theta) = 1 - \cos\theta \ge 0$ for all θ so we have $g(\theta) \ge 0$ for all $\theta \ge 0$. For $f(\theta) = \cos\theta - 1 + \frac{1}{2}\theta^2$ we have $f(0) = 0$ and $f'(\theta) = \theta - \cos\theta = g(\theta) \ge 0$ for all $\theta \ge 0$, and hence $f(\theta) \ge 0$, that is $\cos\theta \ge 1 - \frac{1}{2}\theta^2$ for all $\theta \ge 0$. For $3 \le x < \infty$ we have $0 < \frac{\pi}{x} \le \frac{\pi}{3}$. Since $0 < \frac{\pi}{x}$ we have $\cos\frac{\pi}{x} \ge 1 - \frac{1}{2}\left(\frac{\pi}{x}\right)^2 = \frac{2x^2-\pi^2}{2x^2}$ and hence $\ln\left(\sec\frac{\pi}{x}\right) = -\ln\left(\cos\frac{\pi}{x}\right) \le -\ln\frac{2x^2-\pi^2}{2x^2} = \ln\frac{2x^2}{2x^2-\pi^2}$ so that

$$\int_3^\infty \ln\left(\sec\frac{\pi}{x}\right)dx \le \int_3^\infty \ln\left(\frac{2x^2}{2x^2-\pi^2}\right)dx.$$

We can evaluate this integral by integrating by parts, using $u = \ln\frac{2x^2}{2x^2-\pi^2}$ and $dv = dx$. Note that we have $\frac{du}{dx} = \frac{2x^2-\pi^2}{2x^2}\cdot\frac{(4x)(2x^2-\pi^2)-(2x^2)(4x)}{(2x^2-\pi^2)^2} = \frac{-\pi^2\,dx}{x(2x^2-\pi^2)}$. Also note that by l'Hôpital's Rule we have

$$\lim_{x\to\infty} x\,\ln\frac{2x^2}{2x^2-\pi^2} = \lim_{x\to\infty}\frac{\ln\frac{2x^2}{2x^2-\pi^2}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{-\pi^2}{x(2x^2-\pi^2)}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{\pi^2 x}{2x^2-\pi^2} = 0,$$

and so

$$\begin{aligned}\int_3^\infty \ln\left(\sec\frac{\pi}{x}\right)dx &\le \int_3^\infty \ln\left(\frac{2x^2}{2x^2-\pi^2}\right)dx = \Big[x\,\ln\frac{2x^2}{2x^2-\pi^2}\Big]_3^\infty + \int_3^\infty \frac{\pi^2\,dx}{2x^2-\pi^2}\\ &= \lim_{x\to\infty} x\,\ln\frac{2x^2}{2x^2-\pi^2} - 3\ln\frac{18}{18-\pi^2} + \int_3^\infty \frac{\pi}{\sqrt{2}x-\pi} - \frac{\pi}{\sqrt{2}+\pi}\,dx\\ &= 3\ln\frac{18-\pi^2}{18} + \Big[\frac{\pi}{\sqrt{2}}\ln\frac{\sqrt{2}x-\pi}{\sqrt{2}x+\pi}\Big]_3^\infty = 3\ln\frac{18-\pi^2}{18} + \frac{\pi}{\sqrt{2}}\ln\frac{3\sqrt{2}+\pi}{3\sqrt{2}-\pi} < \infty.\end{aligned}$$

Alternatively, we can argue as follows. As noted above we have $\cos\frac{\pi}{x} \ge 1 - \frac{1}{2}\left(\frac{\pi}{x}\right)^2 = 1 - \frac{\pi^2}{2x^2}$. Since $\ln u$ is concave down with $\ln\frac{1}{3} = -\ln 3$ and $\ln 1 = 0$, the graph of $\ln u$ lies above the line through $\left(\frac{1}{3}, -\ln 3\right)$ and $(1,0)$ so we have $\ln u \ge \frac{2}{3\ln 3}(u-1)$ and hence $-\ln u \le \frac{2}{3\ln 3}(1-u)$. When $3 \le x < \infty$ we have $0 < \frac{\pi^2}{2x^2} \le \frac{\pi^2}{18}$ so that $\frac{1}{3} < 1 - \frac{\pi^2}{18} \le 1 - \frac{\pi^2}{2x^2} < 1$ hence (by taking $u = 1 - \frac{\pi^2}{2x^2}$) we have $-\ln\left(1 - \frac{\pi^2}{2x^2}\right) \le \frac{2}{3\ln 3}\,\frac{\pi^2}{2x^2} = \frac{\pi^2}{3\ln 3}\cdot\frac{1}{x^2}$. Thus $\displaystyle\int_3^\infty \ln\left(\sec\frac{\pi}{x}\right)dx = \int_3^\infty -\ln\left(\cos\frac{\pi}{x}\right)dx \le \int_3^\infty -\ln\left(1-\frac{\pi^2}{2x^2}\right)dx \le \int_3^\infty \frac{\pi^2}{3\ln 3}\cdot\frac{1}{x^2}\,dx = \Big[\frac{\pi^2}{3\ln 3}\cdot\frac{1}{x}\Big]_3^\infty = \frac{\pi^2}{9\ln 3}$.

4: (a) Let $f:[a,b]\to[c,d]$ be bijective and decreasing with $f(a)=d$ and $f(b)=c$. Let $g=f^{-1}:[c,d]\to[a,b]$. Suppose f and g are continuous and consider the area of the region $a\le x\le b$, $c\le y\le f(x)$. Prove that

$$\int_{x=a}^{b}\big(f(x)-c\big)\,dx=\int_{y=c}^{d}\big(g(y)-a\big)\,dy$$

.

Solution: We need to show that

$$\int_{x=a}^{b}f(x)\,dx-\int_{y=c}^{d}g(y)\,dy=\int_{x=a}^{b}c\,dx-\int_{y=c}^{d}a\,dy=c(b-a)-a(d-c)=bc-ad.$$

Let $\epsilon>0$ be arbitrary. Choose $\delta_1>0$ so that for every partition X of $[a,b]$ with $|X|<\delta_1$ we have $\left|S-\int_a^b f\right|<\frac{1}{2}\,\epsilon$ for every Riemann sum S for f on X, and choose $\delta_2>0$ such that for every partition Y of $[c,d]$ with $|Y|<\delta_2$ we have $\left|S-\int_c^d g\right|<\frac{1}{2}\,\epsilon$ for every Riemann sum S for g on Y. Choose a partition X_0 of $[a,b]$ with $|X_0|<\delta_1$ and choose a partition Y_0 of $[c,d]$ with $|Y_0|<\delta_2$. Let $X=X_0\cup g(Y_0)$ and let $Y=Y_0\cup f(X_0)$. Then we have $|X|<\delta_1$ and $|Y|<\delta_2$. Write $X=\{x_0,x_1,\cdots,x_n\}$, whith the x_k in increasing order as usual, and note that, since f is decreasing, we have $Y=\{y_0,y_1,\cdots,y_n\}$ where $y_\ell=f(x_{n-\ell})$ for all ℓ. Since f and g are decreasing, the lower Riemann sums are equal to the sums using the right endpoints. Making the substitution $k=n-\ell$ in one of the sums below and $k=n-\ell+1$ in another, we have

$$\begin{aligned}L(f,X)-L(g,Y)&=\sum_{k=1}^{n}f(x_k)(x_k-x_{k-1}-\sum_{\ell=1}^{n}g(y_\ell)(y_\ell-y_{\ell-1})\\&=\sum_{k=1}^{n}f(x_k)(x_k-x_{k-1})-\sum_{\ell=1}^{n}x_{n-\ell}\big(f(x_{n-\ell})-f(x_{n-\ell+1})\big)\\&=\sum_{k=1}^{n}x_kf(x_k)-\sum_{k=1}^{n}x_{k-1}f(x_k)-\sum_{\ell=1}^{n}x_{n-\ell}f(x_{n-\ell})+\sum_{\ell=1}^{n}x_{n-\ell}f(x_{n-\ell+1})\\&=\sum_{k=1}^{n}x_kf(x_k)-\sum_{k=1}^{n}x_{k-1}f(x_k)-\sum_{k=0}^{n-1}x_kf(x_k)+\sum_{k=1}^{n}x_{k-1}f(x_k)\\&=x_nf(x_n)-x_0f(x_0)x_0\;=\;bc-ad.\end{aligned}$$

By the Triangle Inequality

$$\begin{aligned}\left|\left(\int_a^b f(x)\,dx-\int_b^c g(y)\,dy\right)-(bc-ad)\right|&=\left|\int_a^b f(x)\,dx-\int_b^c g(y)\,dy-L(f,X)+L(g,Y)\right|\\&\le\left|\int_a^b f(x)\,dx-L(f,X)\right|+\left|\int_b^c g(y)\,dy-L(g,Y)\right|<\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2}=\epsilon.\end{aligned}$$

Since $\epsilon>0$ was arbitrary, it follows that $\displaystyle\int_a^b f(x)\,dx-\int_c^d g(y)\,dy=bc-ad$, as required.

(b) Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$. For a partition $X = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$, define

$$\text{Length}(f, X) = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$$

then define the *length* of the graph $y = f(x)$ on $[a, b]$ to be

$$\text{Length}(f) = \sup \Big\{ \text{Length}(f, X) \,\Big|\, X \text{ is a partition for } [a, b] \Big\}$$

(the above supremum can be finite or infinite). We say that f is *rectifiable* on $[a, b]$ when Length(f) is finite. Show that if f is rectifiable on $[a, b]$ then f is integrable on $[a, b]$.

Solution: Suppose that f is rectifiable on $[a, b]$ and let $L = \text{Length}(f, [a, b])$. Suppose, for a contradiction, that f is not integrable on $[a, b]$. Choose $\epsilon > 0$ so that for every partition X of $[a, b]$ we have $U(f, X) - L(f, X) \geq \epsilon$. Let $X = \{x_0, \cdots, x_n\}$ be a partition of $[a, b]$ into equal-sized subintervals of size $\Delta_k x = \frac{b-a}{n} \leq \frac{\epsilon}{3L}$. Choose $t_k \in [x_{k-1}, x_k]$ so $U(f, X) - \sum_{k=1}^{n} f(t_k)\Delta_k x \leq \frac{\epsilon}{3}$ and choose $s_k \in [x_{k-1}, x_k]$ so $\sum_{k=1}^{n} f(s_k)\Delta_k x - L(f, X) \leq \frac{\epsilon}{3}$. Then

$$\begin{aligned}\sum_{k=1}^{n} \big(f(t_k) - f(s_k)\big)\Delta_k x &= \big(U(f, X) - L(f, X)\big) - \big(U(f, X) - \sum_{k=1}^{n} f(t_k)\Delta_k x\big) - \big(\sum_{k=1}^{n} f(s_k)\Delta_k x - L(f, X)\big) \\ &\geq \epsilon - \tfrac{\epsilon}{3} - \tfrac{\epsilon}{3} = \tfrac{\epsilon}{3}.\end{aligned}$$

Now, let Y be the partition $X \cup \{s_1, \cdots, s_n\} \cup \{t_1, \cdots, t_n\}$. Let $u_k = \min\{s_k, t_k\}$ and $v_k = \max\{s_k, t_k\}$ so that $x_{k-1} \leq u_k \leq v_k \leq x_k$ and $Y = \{x_0, u_1, v_1, x_1, u_2, v_2, x_2, \cdots, u_n, v_n, x_n\}$. For $1 \leq k \leq n$, let

$$L_k = \sqrt{(u_k - x_{k-1})^2 + \big(f(u_k) - f(x_{k-1})\big)^2} + \sqrt{(v_k - u_k)^2 + \big(f(v_k) - f(u_k)\big)^2} + \sqrt{(x_k - v_k)^2 + \big(f(x_k) - f(v_k)\big)^2}$$

so that $\text{Length}(f, Y) = \sum_{k=1}^{n} L_k$. Since $L_k \geq \sqrt{(f(v_k) - f(u_k))^2} = \big|f(t_k) - f(s_k)\big| \geq f(t_k) - f(s_k)$, and $\Delta_k x = \frac{b-a}{n}$, and $\frac{n}{b-a} \geq \frac{3L}{\epsilon}$, and $\sum_{k=1}^{n}(f(t_k) - f(s_k))\Delta_k x \geq \frac{\epsilon}{3}$, we have

$$\text{Length}(f, Y) = \sum_{k=1}^{n} L_k \geq \sum_{k=1}^{n} \big(f(t_k) - f(s_k)\big) = \frac{n}{b-a} \sum_{k=1}^{n} \big(f(t_k) - f(s_k)\big)\Delta_k x \geq \frac{3L}{\epsilon} \cdot \frac{\epsilon}{3} = L$$

which is impossible $\big($since $L = \sup\big\{\text{Length}(f, X) \big| X \text{ is a partition}\big\}\big)$.