PMATH 333 Real Analysis, Assignment 5 — Due Mon Nov 18, 11:00 pm

1: For each of the following sequences of functions $(f_n)_{n\geq 1}$, find the set A of points $x \in \mathbb{R}$ for which the sequence of real numbers $\big(f_n(x)\big)_{n\geq 1}$ converges, find the pointwise limit $g(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in A$, and determine whether $f_n \to g$ uniformly in A.

(a) $f_n(x) = (\sin x)^{1/(2n+1)}$

(b) $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$

(c) $f_n(x) = x^n - x^{2n}$

2: Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$, let $(f_n)_{n\geq 1}$ be a sequence of functions $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$, let $g : A \subseteq \mathbb{R} \to \mathbb{R}$ and let $h : \mathbb{R} \to \mathbb{R}$.

(a) Suppose that $\sum_{n\geq 1} a_n$ converges and $\big|f_{n+1}(x) - f_n(x)\big| \leq a_n$ for all $n \geq 1$ and all $x \in A$. Show that $(f_n)_{n\geq 0}$ converges uniformly on A.

(b) Suppose that $f_n \to g$ uniformly on A and $f_n(x) \geq 0$ for all $n \geq 1$ and all $x \in A$. Show that $\sqrt{f_n} \to \sqrt{g}$ uniformly on A.

(c) Suppose that $f_n \to g$ uniformly on A, g is bounded, and h is continuous. Prove that $h \circ f_n \to h \circ g$ uniformly on A.

3: (a) Approximate the value of $e^{3/5}$ so that the absolute error is at most $\frac{1}{1{,}000}$.

(b) Evaluate $\sum\limits_{n=0}^{\infty} \frac{1}{(n+1)(n+2)2^n}$.

(c) Evaluate $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{4^n} \binom{2n}{n}$. Hint: consider $(1+x)^{-1/2}$ and use Abel's Theorem (Part 4 of Theorem 4.23).

4: (a) Show that for $n, m \in \mathbb{Z}^+$ we have

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \left\{ \begin{matrix} 0 \text{ if } n \neq m \\ \pi \text{ if } n = m \end{matrix} \right\} \quad \text{and} \quad \int_{-\pi}^{\pi} x^2 \cos(mx)\,dx = \frac{4(-1)^m}{m^2}\,\pi\,.$$

(b) Suppose that there exists a sequence $(a_n)_{n\geq 1}$ such that $\sum_{n\geq 1} |a_n|$ converges and

$$\sum_{n=1}^{\infty} a_n \cos(nx) = x^2 + c \text{ for all } x \in [-\pi, \pi] \text{ and for some } c \in \mathbb{R}\,.$$

Evaluate the constant c and all of the terms a_n, then evaluate the sums $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

(In fact, such a sequence does exist).