PMATH 333 Real Analysis, Solutions to Assignment 5

1: For each of the following sequences of functions $(f_n)_{n\geq 1}$, find the set A of points $x \in \mathbb{R}$ for which the sequence of real numbers $\big(f_n(x)\big)_{n\geq 1}$ converges, find the pointwise limit $g(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in A$, and determine whether $f_n \to g$ uniformly in A.

(a) $f_n(x) = (\sin x)^{1/(2n+1)}$

Solution: If $x = \pi k$ for some $k \in \mathbb{Z}$ then $\sin x = 0$ so $f_n(x) = 0$ and so $\lim_{n\to\infty} f_n(x) = 0$. If $x \in \big(2\pi k, \pi(2k+1)\big)$ for some $k \in \mathbb{Z}$ then $0 < \sin x \leq 1$ and so $\lim_{n\to\infty} f_n(x) = 1$. If $x \in \big(\pi(2k-1), 2\pi k\big)$ for some $k \in \mathbb{Z}$ then $-1 \leq \sin x < 0$ and so $\lim_{n\to\infty} f_n(x) = -1$. Thus the sequence converges for all $x \in \mathbb{R}$, so $A = \mathbb{R}$, and the limit function $g : \mathbb{R} \to \mathbb{R}$ is given by

$$g(x) = \begin{cases} 0 \text{ , if } x = \pi k \\ 1 \text{ , if } x \in \big(2\pi k, \pi(2k+1)\big) \\ -1 \text{ , if } x \in \big(\pi(2k-1), 2\pi k\big) \end{cases}$$

Since each function f_n is continuous everywhere, but $g(x)$ is not continuous at $x = \pi k$ with $k \in \mathbb{Z}$, the convergence cannot be uniform.

(b) $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$

Solution: For all $x \in \mathbb{R}$ we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sqrt{x^2 + \dfrac{1}{n^2}} = \sqrt{x^2} = |x|$. Thus the set of convergence is $A = \mathbb{R}$ and the limit function $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = |x|$ for all $x \in \mathbb{R}$. The convergence is uniform because given $\epsilon > 0$ we can choose $m > \frac{1}{\epsilon}$ and then for all $x \in \mathbb{R}$ and for all $n \geq m$ we have

$$|f_n(x) - g(x)| = \sqrt{x^2 + \tfrac{1}{n^2}} - \sqrt{x^2} = \frac{\big(x^2 + \frac{1}{n^2}\big) - \big(x^2\big)}{\sqrt{x^2 + \frac{1}{n^2}} + \sqrt{x^2}} = \frac{\frac{1}{n^2}}{\sqrt{x^2 + \frac{1}{n^2}} + \sqrt{x^2}} \leq \frac{\frac{1}{n^2}}{\frac{1}{n}+0} = \tfrac{1}{n} \leq \tfrac{1}{m} < \epsilon.$$

(c) $f_n(x) = x^n - x^{2n}$

Solution: Note that $f_n(x) = x^n - x^{2n} = x^n(1 - x^n)$. When $x < -1$, for even values of n we have $x^n \to +\infty$ and $(1 - x^n) \to -\infty$ so that $f_n(x) = x^n(1 - x^n) \to -\infty$, and for odd values of n we have $x^n \to -\infty$ and $(1 - x^2) \to +\infty$ so that $f_n(x) \to -\infty$, and so $\lim_{n\to\infty} f_n(x) = -\infty$. When $x = -1$, for even values of n we have $f_n(x) = x^n - x^{2n} = 1 - 1 = 0$ and for odd values of n we have $f_n(x) = x^n - x^{2n} = -1 - 1 = -2$ and so $\lim_{n\to\infty} f_n(x)$ does not exist. When $-1 < x < 1$ we have $x^n \to 0$ and $x^{2n} \to 0$ and so $\lim_{n\to\infty} f_n(x) = 0$. When $x = 1$ we have $f_n(x) = 0$ for all n so $\lim_{n\to\infty} f_n(x) = 0$ When $x > 1$ we have $x^n \to \infty$ and $(1 - x^n) \to -\infty$ and so $f_n(x) = x^n(1 - x^n) \to -\infty$. Thus the set of points $x \in \mathbb{R}$ for which the sequence $\big(f_n(x)\big)$ converges is $A = (-1, 1]$ and the limit function $g : (-1, 1] \to \mathbb{R}$ is given by $g(x) = 0$ for all $x \in (-1, 1]$. The convergence is not uniform because given any odd $n \in \mathbb{Z}^+$, since f_n is continuous everywhere with $f_n(-1) = -2$ and $f_n(0) = 0$ we can, by the Intermediate Value Theorem, choose $x \in (-1, 0)$ such that $f_n(x) = -1$ and then we have $|f_n(x) - g(x)| = 1$. Alternatively, we can see that the convergence is not uniform because for all $n \in \mathbb{Z}^+$ we have $f_n\big(\frac{1}{\sqrt[n]{2}}\big) = \frac{1}{\sqrt{2}} - \frac{1}{2} = \frac{\sqrt{2}-1}{2}$ so that $|f_n(x) - g(x)| = \frac{\sqrt{2}-1}{2}$.

2: Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$, let $(f_n)_{n\geq 1}$ be a sequence of functions $f_n: A\subseteq\mathbb{R}\to\mathbb{R}$, let $g:A\subseteq\mathbb{R}\to\mathbb{R}$ and let $h:\mathbb{R}\to\mathbb{R}$.

(a) Suppose that $\sum_{n\geq 1}a_n$ converges and $\big|f_{n+1}(x)-f_n(x)\big|\leq a_n$ for all $n\geq 1$ and all $x\in A$. Show that $(f_n)_{n\geq 0}$ converges uniformly on A.

Solution: Let $\epsilon>0$. Since each $a_n\geq 0$ and $\sum a_n$ converges, by the Cauchy Criterion for Series we can choose $m\in\mathbb{Z}^+$ such that for all $\ell>k\geq m$ we have $\sum\limits_{n=k+1}^{\ell}a_n<\epsilon$. Then for all $\ell>k\geq m$ and all $x\in A$ we have

$$\begin{aligned}\big|f_\ell(x)-f_k(x)\big|&=\big|(f_\ell(x)-f_{\ell-1}(x))+(f_{\ell-1}(x)-f_{\ell-2}(x))+\cdots+(f_{k+1}(x)-f_k(x))\big|\\&\leq\big|f_\ell(x)-f_{\ell-1}(x)\big|+\big|f_{\ell-1}(x)-f_{\ell-2}(x)\big|+\cdots+\big|f_{k+1}(x)-f_k(x)\big|\\&\leq a_\ell+a_{\ell-1}+\cdots+a_{k+1}=\sum_{n=k+1}^{\ell}a_n<\epsilon.\end{aligned}$$

Thus $f_n\to f$ uniformly in A by the Cauchy Criterion for Uniform Convergence of Sequences of Functions.

(b) Suppose that $f_n\to g$ uniformly on A and $f_n(x)\geq 0$ for all $n\geq 1$ and all $x\in A$. Show that $\sqrt{f_n}\to\sqrt{g}$ uniformly on A.

Solution: Let $\epsilon>0$. Since $f_n\to g$ uniformly on A we can choose $m\in\mathbb{Z}^+$ such that for all $n\in\mathbb{Z}^+$, if $n\geq m$ then $|f_n(x)-g(x)|<\epsilon^2$ for all $x\in A$. Let $n\in\mathbb{Z}^+$ with $n\geq m$ and let $x\in A$. If $\sqrt{f_n(x)}+\sqrt{g(x)}<\epsilon$ then (by the Triangle Inequality) $\big|\sqrt{f_n(x)}-\sqrt{g(x)}\big|\leq\sqrt{f_n(x)}+\sqrt{g(x)}<\epsilon$, and if $\sqrt{f_n(x)}+\sqrt{g(x)}\geq\epsilon$ then

$$\big|\sqrt{f_n(x)}-\sqrt{g(x)}\big|=\frac{\big|\sqrt{f_n(x)}-\sqrt{g(x)}\big|\,\big|\sqrt{f_n(x)}+\sqrt{g(x)}\big|}{\big|\sqrt{f_n(x)}+\sqrt{g(x)}\big|}=\frac{\big|f_n(x)-g(x)\big|}{\sqrt{f_n(x)}+\sqrt{g(x)}}<\frac{\epsilon^2}{\epsilon}=\epsilon.$$

Thus $\sqrt{f_n}\to\sqrt{g}$ uniformly on A, as required.

(c) Suppose that $f_n\to g$ uniformly on A, g is bounded, and h is continuous. Prove that $h\circ f_n\to h\circ g$ uniformly on A.

Solution: Since g is bounded we can choose $M\geq 0$ so that $|g(x)|\leq M$ for all $x\in A$. Since $f_n\to g$ uniformly on A we can choose $m_1\in\mathbb{Z}^+$ such that $n\geq m_1\Longrightarrow\big|f_n(x)-g(x)\big|\leq 1$ for all $x\in A$. Then for $n\geq m_1$ and $x\in A$ we have $\big|f_n(x)\big|\leq\big|f_n(x)-g(x)\big|+\big|g(x)\big|\leq 1+M$ so that $f_n(x)\in\big[-(M+1),M+1\big]$. Let $\epsilon>0$. Since h is uniformly continuous on $\big[-(M+1),M+1\big]$, we can choose $\delta>0$ so that for all $u,v\in\big[-(M+1),M+1\big]$ we have $|u-v|<\delta\Longrightarrow\big|h(u)-h(v)\big|<\epsilon$. Since $f_n\to g$ uniformly on A we can choose $m\geq m_1$ so that $n\geq m\Longrightarrow\big|f_n(x)-g(x)\big|<\delta$ for all $x\in A$. Let $n\geq m$ and let $x\in A$. Then we have $f_n(x),g(x)\in\big[-(M+1),M+1\big]$ with $\big|f_n(x)-g(x)\big|<\delta$ and hence $\big|h\big(f_n(x)\big)-h\big(g(x)\big)\big|<\epsilon$.

3: (a) Approximate the value of $e^{3/5}$ so that the absolute error is at most $\frac{1}{1{,}000}$.

Solution: We have

$$\begin{aligned} e^{3/5} &= 1 + \left(\tfrac{3}{5}\right) + \tfrac{1}{2!}\left(\tfrac{3}{5}\right)^2 + \tfrac{1}{3!}\left(\tfrac{3}{5}\right)^3 + \tfrac{1}{4!}\left(\tfrac{3}{5}\right)^4 + \tfrac{1}{5!}\left(\tfrac{3}{5}\right)^5 \cdots \\ &\cong 1 + \left(\tfrac{3}{5}\right) + \tfrac{1}{2!}\left(\tfrac{3}{5}\right)^2 + \tfrac{1}{3!}\left(\tfrac{3}{5}\right)^3 + \tfrac{1}{4!}\left(\tfrac{3}{5}\right)^4 \\ &= 1 + \tfrac{3}{5} + \tfrac{9}{50} + \tfrac{9}{250} + \tfrac{27}{5000} = \tfrac{9107}{5000} = 1.8214 \end{aligned}$$

with error

$$\begin{aligned} E &= \tfrac{1}{5!}\left(\tfrac{3}{5}\right)^5 + \tfrac{1}{6!}\left(\tfrac{3}{5}\right)^6 + \tfrac{1}{7!}\left(\tfrac{3}{5}\right)^7 + \tfrac{1}{8!}\left(\tfrac{3}{5}\right)^8 + \cdots \\ &= \tfrac{1}{5!}\left(\tfrac{3}{5}\right)^5 \left(1 + \tfrac{1}{6}\left(\tfrac{3}{5}\right) + \tfrac{1}{6\cdot 7}\left(\tfrac{3}{5}\right)^2 + \tfrac{1}{6\cdot 7\cdot 8}\left(\tfrac{3}{5}\right)^3 + \cdots\right) \\ &\le \tfrac{1}{5!}\left(\tfrac{3}{5}\right)^5 \left(1 + \tfrac{1}{10} + \tfrac{1}{10^2} + \tfrac{1}{10^3} + \cdots\right) \\ &= \frac{\frac{1}{5!}\left(\frac{3}{5}\right)^5}{1 - \frac{1}{10}} = \tfrac{1}{5!}\cdot\tfrac{3^5}{5^5}\cdot\tfrac{10}{9} = \tfrac{3^2}{2^2\,5^5} = \tfrac{9}{12500} < \tfrac{1}{1000} \end{aligned}$$

by the Comparison Test and the formula for the sum of a geometric series.

(b) Evaluate $\sum\limits_{n=0}^{\infty} \frac{1}{(n+1)(n+2)2^n}$.

Solution: Starting with a basic geometric series and integrating twice (using Theorem 4.31) then dividing by x^2, for $0 \ne |x| < 1$ we have

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \\ -\ln(1-x) &= x + \tfrac{1}{2}\,x^2 + \tfrac{1}{3}\,x^3 + \tfrac{1}{4}\,x^4 + \cdots \\ (1-x)\ln(1-x) + x &= \tfrac{1}{1\cdot 2}\,x^2 + \tfrac{1}{2\cdot 3}\,x^3 + \tfrac{1}{3\cdot 4}\,x^4 + \tfrac{1}{4\cdot 5}\,x^5 + \cdots \\ \frac{(1-x)\ln(1-x) + x}{x^2} &= \tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3}\,x + \tfrac{1}{3\cdot 4}\,x^2 + \tfrac{1}{4\cdot 5}\,x^3 + \cdots \quad = \sum_{n=1}^{\infty} \tfrac{1}{(n+1)(n+2)}\,x^n\,. \end{aligned}$$

Put in $x = \frac{1}{2}$ to get

$$\sum_{n=0}^{\infty} \tfrac{1}{(n+1)(n+2)2^n} = \frac{\frac{1}{2}\ln\frac{1}{2} + \frac{1}{2}}{\frac{1}{4}} = 2 - 2\ln 2.$$

(c) Evaluate $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{4^n}\binom{2n}{n}$.

Solution: Let $a_n = \frac{(-1)^n}{4^n}\binom{2n}{n}$. For $n \ge 1$ we have

$$|a_n| = \tfrac{1}{4^n}\binom{2n}{n} = \tfrac{(2n)!}{(2^n n!)^2} = \tfrac{1\cdot 2\cdot 3\cdot \ldots \cdot 2n}{(2\cdot 4\cdot 6\cdot \ldots \cdot 2n)^2} = \tfrac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n}.$$

Since $a_0 = 1$ and $|a_n| = \frac{2n-1}{2n}\,|a_{n-1}| \le |a_{n-1}|$ for $n \ge 1$, it follows that the sequence $\big(|a_n|\big)$ is decreasing. Since

$$|a_n|^2 = \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{5}{6}\cdot\ldots\cdot\tfrac{2n-1}{2n}\cdot\tfrac{2n-1}{2n} \le \tfrac{1}{2}\cdot\tfrac{2}{3}\cdot\tfrac{3}{4}\cdot\tfrac{4}{5}\cdot\ldots\cdot\tfrac{2n-1}{2n}\cdot\tfrac{2n}{2n+1} = \tfrac{1}{2n+1}$$

we have $|a_n| \le \frac{1}{\sqrt{2n+1}} \longrightarrow 0$ as $n \to \infty$. Thus $\sum a_n = \sum (-1)^n |a_n|$ converges by the Alternating Series Test. Note that

$$\tfrac{(-1)^n}{4^n}\binom{2n}{n} = \tfrac{(-1)^n}{4^n}\cdot\tfrac{(2n)!}{(n!)^2} = (-1)^n\,\tfrac{1\cdot 2\cdot 3\cdots(2n)}{(2\cdot 4\cdot 6\cdots(2n))^2} = (-1)^n\,\tfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} = \tfrac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2n-1}{2})}{n!} = \binom{-1/2}{n}$$

so for $|x| < 1$, by Theorem 4.40 (the sum of the binomial series), we have

$$(1+x)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}x^n = \sum_{n=0}^{n}\tfrac{(-1)^n}{4^n}\binom{2n}{n}x^n.$$

Since $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}$ converges, it follows from Abel's Theorem (Part 4 of Theorem 4.23) that $\sum\limits_{n=0}^{n}\frac{(-1)^n}{4^n}\binom{2n}{n}x^n$ converges uniformly on $[0,1]$ and hence, by Theorem 4.14 (uniform convergence and continuity), its sum $g(x) = \sum\limits_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}x^n$ is continuous on $[0,1]$. Since $f(x) = (1+x)^{-1/2}$ is also continuous on $[0,1]$ with $f(x) = g(x)$ when $x < 1$, we have $g(1) = f(1)$, that is

$$\sum_{n=0}^{\infty}\tfrac{(-1)^n}{4^n}\binom{2n}{n} = g(1) = f(1) = (1+1)^{-1/2} = \tfrac{1}{\sqrt{2}}\,.$$

4: (a) Show that for $n, m \in \mathbb{Z}$ with $n, m \geq 1$ we have

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 0 \text{ if } n \neq m \\ \pi \text{ if } n = m \end{cases} \qquad \text{and} \qquad \int_{-\pi}^{\pi} x^2\,\cos(mx)\,dx = \tfrac{4(-1)^m}{m^2}\,\pi\,.$$

Solution: When $n = m$, the first integral becomes

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \int_{-\pi}^{\pi} \tfrac{1}{2} + \tfrac{1}{2}\cos(2nx)\;dx = \Big[\tfrac{1}{2}\theta + \tfrac{1}{4n}\sin(2nx)\Big]_{-\pi}^{\pi} = \pi.$$

When $n \neq m$, using the trigonometric identity $\cos(a)\cos(b) = \frac{1}{2}\big(\cos(a+b) + \cos(a-b)\big)$, the first integral becomes

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx &= \int_{-\pi}^{\pi} \tfrac{1}{2}\cos((n+m)x) + \tfrac{1}{2}\cos((n-m)x)\;dx \\ &= \Big[\tfrac{1}{2(n+m)}\sin((n+m)x) + \tfrac{1}{2(n-m)}\sin((n-m)x)\Big]_{-\pi}^{\pi} = 0. \end{aligned}$$

Integrating by parts twice, first using $u = x^2$, $du = 2x\,dx$, $v = \frac{1}{m}\sin mx$ and $dv = \cos mx\;dx$, then using $u = \frac{2}{m}x$, $du = \frac{2}{m}\,dx$, $v = -\frac{1}{m}\cos mx$ and $dv = \sin mx\;dx$, the second integral becomes

$$\begin{aligned} \int_{-\pi}^{\pi} x^2\cos(mx)\,dx &= \Big[\tfrac{1}{m}\,x^2\sin(mx) - \int \tfrac{2}{m}\,x\,\sin(mx)\,dx\Big]_{-\pi}^{\pi} \\ &= \Big[\tfrac{1}{m}\,x^2\sin(mx) + \tfrac{2}{m^2}\,x\cos(mx) - \int \tfrac{2}{m^2}\cos(mx)\,dx\Big]_{-\pi}^{\pi} \\ &= \Big[\tfrac{1}{m}\,x^2\sin(mx) + \tfrac{2}{m^2}\,x\cos(mx) - \tfrac{2}{m^2}\,\sin(mx)\Big]_{-\pi}^{\pi} \\ &= \big(\tfrac{2}{m^2}\,\pi\cos(mx)\big) - \big(\tfrac{2}{m^2}(-\pi)\cos(-\pi x)\big) = \tfrac{4\pi}{m^2}\cos(mx) = \tfrac{4\pi(-1)^m}{m^2}. \end{aligned}$$

(b) Suppose that there exists a sequence $\{a_n\}$ such that $\sum |a_n|$ converges which has the property that

$$\sum_{n=1}^{\infty} a_n\,\cos(nx) = x^2 + c \text{ for all } x \in [-\pi, \pi] \text{ and for some } c \in \mathbb{R}\,.$$

Evaluate the constant c and all of the terms a_n, then evaluate the sums $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

Solution: Note that since $\sum |a_n|$ converges, the series $\sum a_n\cos(nx)$ and $\sum a_n\cos(nx)\cos(mx)$ both converge uniformly by the Weirstrass M-Test.

We have $\displaystyle\int_{-\pi}^{\pi} x^2 + c\,dx = \Big[\tfrac{1}{3}\,x^3 + cx\Big]_{-\pi}^{\pi} = \tfrac{2}{3}\pi^3 + 2\pi c$, and from uniform convergence we also have

$$\int_{-\pi}^{\pi} x^2 + c\,dx = \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n\cos(nx)\,dx = \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos(nx)\,dx = 0.$$

Thus $\frac{2}{3}\pi^3 + 2\pi c = 0$ and so $c = -\frac{\pi^2}{3}$.

Also, for each m we have $\displaystyle\int_{-\pi}^{\pi}(x^2+c)\cos(mx)\,dx = \tfrac{4(-1)^m}{m^2}\,\pi$ by part (a), since $\displaystyle\int_{-\pi}^{\pi}\cos(mx)\,dx = 0$, and from uniform convergence, we also have

$$\int_{-\pi}^{\pi}(x^2+c)\cos(mx)\,dx = \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n\cos(nx)\cos(mx)\,dx = \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx = a_m\,\pi\,.$$

Thus $\frac{4(-1)^m}{m^2}\,\pi = a_m\,\pi$ and so $a_m = \frac{4(-1)^m}{m^2}$.

For all $x \in [-\pi, \pi]$, we have $x^2 - \frac{\pi^2}{3} = \sum\limits_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)$. Put in $x = \pi$ to get $\pi^2 - \frac{\pi^2}{3} = \sum\limits_{n=1}^{\infty} \frac{4(-1)^n}{n^2}(-1)^n$ and so $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Put in $x = 0$ to get $-\frac{\pi^2}{3} = \sum\limits_{n=1}^{\infty} \frac{4(-1)^n}{n^2}$ and so $\sum\limits_{n=1}^{\infty} \frac{(-1)^2}{2^2} = \frac{\pi^2}{12}$.