PMATH 333 Real Analysis, Assignment 6 Due Mon Dec 2, 11:00 pm

1: (a) Let $A=\left\{x\in\mathbb{R}^2\middle|0<|x|\leq 1\right\}$. Prove, from the definition of a compact set, that A is not compact.

(b) Let $B=\left\{(x,y)\in\mathbb{R}^2\,\middle|\,y>x^2\right\}$. Prove, from the definition of an open set, that B is open in $\mathbb{R}^2$.

(c) For $n\geq 1$, let $s_n=\sum\limits_{k=1}^{n}\left(\frac{1+i}{3}\right)^k$. Prove, from the definition of a limit, that $\lim\limits_{n\to\infty}s_n=\frac{1+3\,i}{5}$.

2: For each of the following subsets $A\subseteq\mathbb{R}^n$, determine whether A is closed, whether A is compact, and whether A is connected.

(a) $A=\left\{(t^2-1\,,\,t^3-t)\in\mathbb{R}^2\,\middle|\,t\in\mathbb{R}\right\}$.

(b) $A=\left\{(0,0)\neq(x,y)\in\mathbb{R}^2\,\middle|\,\left|\mathrm{Re}\left(\frac{1}{x+iy}\right)\right|\geq 1\right\}$ (where $\mathrm{Re}(z)$ denotes the real part of $z\in\mathbb{C}$).

(c) $A=\left\{(u,v,w,x,y,z)\in\mathbb{R}^6\,\middle|\,\mathrm{rank}\begin{pmatrix}u&v&w\\x&y&z\end{pmatrix}\neq 2\right\}$.

3: (a) Prove that if the sets $A,B\subseteq\mathbb{R}^n$ are connected and $A\cap B\neq\emptyset$, then $A\cup B$ is connected.

(b) Prove that if $A\subseteq\mathbb{R}^n$ and $B\subseteq\mathbb{R}^m$ are both connected then $A\times B\subseteq\mathbb{R}^{n+m}$ is connected.

(c) Prove that if $A\subseteq\mathbb{R}^n$ and $B\subseteq\mathbb{R}^m$ are both compact then $A\times B\subseteq\mathbb{R}^{n+m}$ is compact.

4: Let $\mathbb{R}^\omega$ be the set of all sequences in $\mathbb{R}$, that is $\mathbb{R}^\omega=\left\{x=(x_j)_{j\geq1}\middle|\text{each } x_j\in\mathbb{R}\right\}$ and let $\mathbb{R}^\infty$ be the set of eventually zero sequences in $\mathbb{R}$, that is $\mathbb{R}^\infty=\left\{x=(x_j)_{j\geq1}\in\mathbb{R}^\omega\,\middle|\exists\,m\in\mathbb{Z}^+\ \forall\,j\in\mathbb{Z}^+\,(j\geq m\implies x_j=0)\right\}$. For $x,y\in\mathbb{R}^\infty$, define $x\bullet y=\sum_{j=1}^\infty x_jy_j$ and $|x|=\left(x\bullet x\right)^{1/2}$.

When $(x_n)_{n\geq1}$ is a sequence in $\mathbb{R}^\infty$, each $x_n\in\mathbb{R}^\infty$, and we can write $x_n=(x_{n,j})_{j\geq1}=(x_{n,1},x_{n,2},x_{n,3},\cdots)$. For a sequence $(x_n)_{n\geq1}$ in $\mathbb{R}^\infty$ and an element $a\in\mathbb{R}^\infty$, we say the sequence $(x_n)_{n\geq1}$ *converges* to a in $\mathbb{R}^\infty$, and we write $x_n\to a$ in $\mathbb{R}^\infty$ or $\lim\limits_{n\to\infty}x_n=a$ in $\mathbb{R}^\infty$, when $\forall\,\epsilon>0\ \exists\,m\in\mathbb{Z}^+\ \forall\,n\in\mathbb{Z}^+\ (n\geq m\implies|x_n-a|<\epsilon)$, we say that $(x_n)_{n\geq1}$ is *bounded* when $\exists\,r\geq0\ \forall\,n\in\mathbb{Z}^+\ |x_n|\leq r$, and we say that $(x_n)_{n\geq1}$ is *Cauchy* when $\forall\,\epsilon>0\ \exists\,m\in Z^+\ \forall\,k,\ell\in\mathbb{Z}^+\,(k,\ell\geq m\implies|x_k-x_\ell|<\epsilon)$.

(a) Prove that for all sequences $(x_n)_{n\geq1}$ in $\mathbb{R}^\infty$, and all $a\in\mathbb{R}^\infty$, if $\lim\limits_{n\to\infty}x_n=a$ in $\mathbb{R}^\infty$ then $\lim\limits_{n\to\infty}x_{n,j}=a_j$ for all $j\in\mathbb{Z}^+$, but that the converse does not hold.

(b) Prove that for all sequences $(x_n)_{n\geq1}$ in $\mathbb{R}^\infty$, if the sequence $(x_n)_{n\geq1}$ converges in $\mathbb{R}^\infty$ (to some $a\in\mathbb{R}^\infty$) then it is Cauchy, but that the converse does not hold.

(c) Determine whether every bounded sequence $(x_n)_{n\geq1}$ in $\mathbb{R}^\infty$ has a convergent subsequence $(x_{n_k})_{k\geq1}$ in $\mathbb{R}^\infty$.