PMATH 333 Real Analysis, Solutions to Assignment 6

1: (a) Let $A=\big\{x\in\mathbb{R}^2\big|0<|x|\leq 1\big\}$. Prove, from the definition of a compact set, that A is not compact.

Solution: For each $k\in\mathbb{Z}^+$ let U_k be the open set $U_k=\overline{B}\big(0,\frac{1}{k}\big)^c=\big\{x\in\mathbb{R}^n\big||x|>\frac{1}{k}\big\}$ and let $S=\big\{U_k\big|k\in\mathbb{Z}^+\big\}$. Note that $\bigcup S=\mathbb{R}^n\setminus\{0\}$ so S is an open cover of A. Let T be any finite subset of S. If $T=\emptyset$ then $\bigcup T=\emptyset$ so $A\not\subseteq\bigcup T$. Suppose that $T\neq\emptyset$, say $T=\big\{U_{k_1},U_{k_2},\cdots,U_{k_m}\big\}$ with $k_1<k_2<\cdots<k_m$. Since $U_{k_1}\subseteq U_{k_2}\subseteq\cdots\subseteq U_{k_m}$ we have $\bigcup T=\bigcup_{i=1}^m U_{k_i}=U_{k_m}=\overline{B}\big(0,\frac{1}{k_m}\big)^c$ and so $A\not\subseteq\bigcup T$. This shows that the open cover S has no finite subcover T, and so A is not compact.

(b) Let $B=\big\{(x,y)\in\mathbb{R}^2\,\big|\,y>x^2\big\}$. Show, from the definition of an open set, that B is open in $\mathbb{R}^2$.

Solution: Let $(a,b)\in B$ so we have $b>a^2$ and hence $\sqrt{b}>|a|$. Let $r=\min\Big(\frac{b-a^2}{2},\,\frac{\sqrt{b}-|a|}{2}\Big)$. We claim that $B\big((a,b),r\big)\subseteq B$. Let $(x,y)\in B\big((a,b),r\big)$. Note that

$$|x-a|\leq\sqrt{(x-a)^2+(y-b)^2}=d\big((a,b),(x,y)\big)<r\leq\tfrac{\sqrt{b}-|a|}{2}$$

and similarly

$$|y-b|<r\leq\tfrac{b-a^2}{2}.$$

It follows that $|x|-|a|\leq|x-a|<\frac{\sqrt{b}-|a|}{2}$ so that $|x|\leq\frac{\sqrt{b}+|a|}{2}$ and that $b-y\leq|y-b|<\frac{b-a^2}{2}$ so that $y>\frac{b+a^2}{2}$. Note that $0\leq\big(\sqrt{b}-|a|\big)^2=b+a^2-2|a|\sqrt{b}$ so we have $2|a|\sqrt{b}\leq b+a^2$. It follows that

$$x^2<\Big(\tfrac{\sqrt{b}+|a|}{2}\Big)^2=\tfrac{b+a^2+2|a|\sqrt{b}}{4}\leq\tfrac{b+a^2}{2}<y.$$

Since $y>x^2$ we have $(x,y)\in B$. This shows that $B\big((a,b),r\big)\subseteq B$, as claimed, and so S is open.

(c) For $n\geq 1$, let $s_n=\sum\limits_{k=1}^{n}\big(\frac{1+i}{3}\big)^k$. Prove, from the definition of a limit, that $\lim\limits_{n\to\infty}s_n=\frac{1+3\,i}{5}$.

Solution: First note that $\mathbb{C}=\mathbb{R}^2$ (when $x,y\in\mathbb{R}$, the ordered pair $(x,y)\in\mathbb{R}$ is equal to the complex number $z=x+iy\in\mathbb{C}$), and the usual norm in $\mathbb{C}$ is equal to the usual norm in $\mathbb{R}^2$: for $z=x+iy=(x,y)$ we have $|z|=\sqrt{x^2+yy^2}=\big|(x,y)\big|$. From the formula for the sum of a geometric series, or by noting that

$$s_n\big(1-\tfrac{1+i}{3}\big)=\sum_{k=1}^{n}\big(\tfrac{1+i}{3}\big)^k-\sum_{k=2}^{n+1}\big(\tfrac{1+i}{3}\big)^k=\big(\tfrac{1+i}{3}\big)-\big(\tfrac{1+i}{3}\big)^{n+1},$$

we have

$$s_n=\frac{\big(\frac{1+i}{3}\big)-\big(\frac{1+i}{3}\big)^{n+1}}{1-\frac{1+i}{3}}=\frac{\big(\frac{1+i}{3}\big)\big(1-\big(\frac{1+i}{3}\big)^n\big)}{\frac{2-i}{3}}=\frac{(1+i)(2+i)\big(1-\big(\frac{1+i}{3}\big)^n\big)}{(2-i)(2+i)}=\tfrac{1+3i}{5}\big(1-\big(\tfrac{1+i}{3}\big)^n\big)=\tfrac{1+3i}{5}-\tfrac{1+3i}{5}\big(\tfrac{1+i}{3}\big)^n$$

and hence

$$\big|s_n-\tfrac{1+3i}{5}\big|=\big|\tfrac{1+3i}{5}\,\big(\tfrac{1+i}{3}\big)^n\big|=\big|\tfrac{1+3i}{5}\big|\,\big|\tfrac{1+i}{3}\big|^n=\tfrac{\sqrt{10}}{5}\big(\tfrac{\sqrt{2}}{3}\big)^n.$$

It follows that $\lim\limits_{n\to\infty}s_n=\frac{1+3i}{5}$: indeed given $\epsilon>0$, since $\frac{\sqrt{2}}{3}<1$ we can choose $m\in\mathbb{Z}^+$ so that $\big(\frac{\sqrt{2}}{3}\big)^m<\frac{\epsilon}{\sqrt{10}/5}$, and then when $n\geq m$ we have

$$\big|s_n-\tfrac{1+3i}{5}\big|=\tfrac{\sqrt{10}}{5}\big(\tfrac{\sqrt{2}}{3}\big)^n\leq\tfrac{\sqrt{10}}{5}\big(\tfrac{\sqrt{3}}{2}\big)^m<\epsilon.$$

2: For each of the following subsets $A\subseteq\mathbb{R}^n$, determine whether A is closed, whether A is compact, and whether A is connected.

(a) $A=\left\{(t^2-1\,,\,t^3-t)\in\mathbb{R}^2\,\middle|\,t\in\mathbb{R}\right\}$.

Solution: Note that $A=f(\mathbb{R})$ where $f:\mathbb{R}\to\mathbb{R}^2$ is given by $(x,y)=f(t)=\big(t^2-1\,,\,t^3-t\big)$. Since $\mathbb{R}$ is connected and f is continuous, it follows that $A=f(\mathbb{R})$ is connected.

We claim that $A=g^{-1}(0)$ where $g:\mathbb{R}^2\to\mathbb{R}$ is given by $g(x,y)=x^3+x^2-y^2$. Let $(x,y)\in A$, say $(x,y)=(t^2-1\,,\,t^3-t)$. Then $x^3+x^2=(t^6-3t^4+3t^2-1)+(t^4-2t^2+1)=t^6-2t^4+t^2=(t^3-t)^2=y^2$ so that $g(x,y)=0$. This shows that $A\subseteq g^{-1}(0)$. Now let $(x,y)\in g^{-1}(0)$, so we have $y^2=x^3+x^2$. If $x=0$ then $y^2=x^3+x^2=0$ so that $y=0$, and in this case we can choose $t=1$ to get $t^2-1=0=x$ and $t^3-t=0=y$ so that $(x,y)\in A$. If $x\neq 0$ then we can choose $t=\frac{y}{x}$ to get $t^2-1=\frac{y^2}{x^2}-1=\frac{y^2-x^2}{x^2}=\frac{x^3}{x^2}=x$ and $t^3-t=t(t^2-1)=\frac{y}{x}\cdot x=y$ so that again $(x,y)\in A$. This shows that $g^{-1}(0)\subseteq A$, and hence $A=g^{-1}(0)$, as claimed. Since $\{0\}$ is closed and g is continuous, it follows that $A=g^{-1}(\{0\})$ is closed.

Finally, we note that A is not compact because A is not bounded: indeed given any $M\geq 0$ we can choose $t\geq 1$ such that $t^2>M+1$, and then $(x,y)=(t^2-1\,,\,t^3-t)\in A$ with $\big|(x,y)\big|=\big|(t^2-1\,,\,t^3-t)\big|\geq t^2-1>M$.

(b) $A=\left\{(0,0)\neq(x,y)\in\mathbb{R}^2\,\middle|\,\left|\mathrm{Re}\big(\frac{1}{x+iy}\big)\right|\geq 1\right\}$ (where $\mathrm{Re}(z)$ denotes the real part of $z\in\mathbb{C}$).

Solution: For $a,b\in\mathbb{R}$ with $(a,b)\neq(0,0)$ we have $\frac{1}{a+ib}=\frac{a-ib}{a^2+b^2}$ so

$$\begin{aligned}\left|\mathrm{Re}\big(\tfrac{1}{a+ib}\big)\right|\geq 1 &\iff |a|\geq a^2+b^2\big(a^2+b^2\leq a \text{ or } a^2+b^2\leq -a\big)\\ &\iff \big(a-\tfrac12\big)^2+b^2\leq\tfrac14 \text{ or } \big(a+\tfrac12\big)^2+b^2\leq\tfrac14.\end{aligned}$$

Thus $A=(B\cup C)\setminus\{(0,0)\}$ where B and C are the closed balls of radius $\frac12$ centered at $\big(\frac12,0\big)$ and $\big(-\frac12,0\big)$. This set A is not closed since $(0,0)\notin A$ but $(0,0)$ is a limit point of A (indeed for $x_n=\big(\frac1n,0\big)$ we have $x_n\in A$ and $x_n\to(0,0)$). Since A is not closed, it not compact. Also, A is not connected since it can be separated by the disjoint open sets $U=\big\{(x,y)\big|x>0\big\}$ and $V=\big\{(x,y)\big|x<0\big\}$.

(c) $A=\left\{(u,v,w,x,y,z)\in\mathbb{R}^6\,\middle|\,\mathrm{rank}\begin{pmatrix}u&v&w\\x&y&z\end{pmatrix}\neq 2\right\}$.

Solution: Note that we have $\mathrm{rank}\begin{pmatrix}u&v&w\\x&y&z\end{pmatrix}=2$ if and only if some pair of columns is linearly independent if and only if one of the three 2×2 submatrices $\begin{pmatrix}u&v\\x&y\end{pmatrix}$, $\begin{pmatrix}u&w\\x&z\end{pmatrix}$ and $\begin{pmatrix}v&w\\y&z\end{pmatrix}$ is invertible if and only if one of the three determinants $uy-vx$, $uz-wx$ and $vz-wy$ is non-zero. Thus we have

$$\mathrm{rank}\begin{pmatrix}u&v&w\\x&y&z\end{pmatrix}\neq 2\iff\Big(uy-vx=0 \text{ and } uz-wx=0 \text{ and } vz-wy=0\Big)$$

and hence

$$A=f^{-1}\big(\{0\}\big)\cap g^{-1}\big(\{0\}\big)\cap h^{-1}\big(\{0\}\big)$$

where $f,g,h:\mathbb{R}^6\to\mathbb{R}$ are given by

$$\begin{aligned}f(u,v,w,x,y,z)&=uy-vx\,,\\ g(u,v,w,x,y,z)&=uz-wx\,,\\ h(u,v,w,x,y,z)&=vz-wy\,.\end{aligned}$$

Since f, g and h are continuous (they are polynomials) and $\{0\}$ is closed in $\mathbb{R}$, it follows (from Theorem 5.29) that the sets $f^{-1}\big(\{0\}\big)$, $g^{-1}\big(\{0\}\big)$ and $h^{-1}\big(\{0\}\big)$ are all closed, and hence the set A is closed (by Theorem 4.36). On the other hand, A is not bounded because for $e_1=(1,0,0,0,0,0)$ we have $re_1\in A$ for all $r\in\mathbb{R}$ and $\|re_1\|=|r|$. Since A is not bounded, it is not compact (by Theorem 6.21). Finally, we note that A is path-connected (hence connected), indeed given $a,b\in A$, the map $\alpha:[0,1]\subseteq\mathbb{R}\to\mathbb{R}^6$ given by

$$\alpha(t)=\left\{\begin{array}{l}(1-2t)a \text{ for } 0\leq t\leq\frac12\\ (2t-1)b \text{ for } \frac12\leq t\leq 1\end{array}\right\}$$

is continuous with $\alpha(0)=a$, $\alpha\big(\frac12\big)=0$ and $\alpha(1)=b$, and we have $\alpha(t)\in A$ for all t (because when X is a matrix and $r\in\mathbb{R}$, we have $\mathrm{rank}(rX)=\mathrm{rank}(X)$ when $r\neq 0$, and we have $\mathrm{rank}(rX)=0$ when $r=0$).

3: (a) Prove that if the sets $A, B \subseteq \mathbb{R}^n$ are connected and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

Solution: Suppose that A and B are connected in $\mathbb{R}^n$ and that $A \cap B \neq \emptyset$. Choose $c \in A \cap B$. Suppose, for a contradiction, that $A \cup B$ is disconnected. Choose open sets U and V in $\mathbb{R}^n$ which separate $A \cup B$ (that is, $U \cap (A \cup B) \neq \emptyset$, $V \cap (A \cup B) \neq \emptyset$, $U \cup V = \emptyset$, and $A \cup B \subseteq U \cup V$). Since $c \in A \cap B \subseteq A \cup B \subseteq U \cup V$, either $c \in U$ or $c \in V$. By interchanging U and V if necessary, we can suppose that $c \in U$. Note that since $c \in A$ and $c \in U$ and A is connected, it follows that $A \subseteq U$ because if we had $A \not\subseteq U$ then (since $A \subseteq U \cup V$) we would have $A \cap V \neq \emptyset$, and then U and V would separate A (since $c \in U \cap A$ so $U \cap A \neq \emptyset$, and $U \cap V = \emptyset$, and $A \subseteq A \cup B \subseteq U \cup V$). Similarly, since $c \in B$ and $c \in U$ and B is connected, it follows that $B \subseteq U$. Since $A \subseteq U$ and $B \subseteq U$, we have $A \cup B \subseteq U$. Since $A \cup B \subseteq U$ and $U \cap V = \emptyset$, we must have $V \cap (A \cup B) = \emptyset$, which contradicts the fact that U and V separate $A \cup B$.

(b) Prove that if $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are both connected then $A \times B \subseteq \mathbb{R}^{n+m}$ is connected.

Solution: Suppose that A and B are connected. Suppose for a contradiction that $A \times B$ is disconnected. Choose open sets U and V in $\mathbb{R}^{n+m}$ which separate U and V. Choose $(a, b) \in U \cap (A \times B)$ and $(b, c) \in V \cap (A \times B)$. Since A is connected and the function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^{n+m}$ given by $f(x) = (x, b)$ is continuous, it follows that the set $f(A) = A \times \{b\}$ is connected. Since $A \times \{b\}$ is connected and $(a, b) \in U \cap (A \times \{b\}$, it follows that we must have $A \times \{b\} \subseteq U$ (otherwise the sets U and V would separate $A \times \{b\}$). In particular, we have $(c, b) \in A \times \{b\} \subseteq U$. Since B is connected and the map $g : B \subseteq \mathbb{R}^m \to \mathbb{R}^{n+m}$ given by $g(y) = (c, y)$ is continuous, it follows that the set $f(B) = \{c\} \times B$ is connected. Since $\{c\} \times B$ is connected and $(c, b) \in U \cap (\{c\} \times B)$, it follows that $\{c\} \times B \subseteq U$ (otherwise the sets U and V would separate $\{c\} \times B$). In particular, we have $(c, d) \in \{c\} \times B \subseteq U$. But this is not possible since $(c, d) \in V$ and $U \cap V = \emptyset$.

(c) Prove that if $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are both compact then $A \times B \subseteq \mathbb{R}^{n+m}$ is compact.

Solution: We shall use the Sequential Characterization of Cmpactness. Suppose that A and B are compact. Let $(x_n, y_n)_{n \geq 1}$ be a sequence in $A \times B$. Since A is compact, the sequence $(x_n)_{n \geq 1}$ has a subsequence which converges to an element in A. Let $(x_{n_k})_{k \geq 1}$ be a subsequence with $x_{n_k} \to a \in A$. Since B is compact, the sequence $(y_{n_k})_{k \geq 1}$ has a subsequence which converges to an element in B. Let $(y_{n_{k_j}})_{j \geq 1}$ be a subsequence with $y_{n_{k_j}} \to b \in B$. Since $(x_{n_{k_j}})_{j \geq 1}$ is a subsequence of $(x_{n_k})_{k \geq 1}$ and $x_{n_k} \to a \in A$, we also have $x_{n_{k_j}} \to a \in (a, b)$ and so $(x_{n_{k_j}}, y_{n_{k_j}}) \to (a, b)$. This shows that every sequence in $A \times B$ has a subsequence which converges to an element in $A \times B$, and hence $A \times B$ is compact.

4: Let $\mathbb{R}^\omega$ be the set of all sequences in $\mathbb{R}$, that is $\mathbb{R}^\omega = \big\{x = (x_j)_{j\geq 1}\big|\,\text{each } x_j \in \mathbb{R}\big\}$ and let $\mathbb{R}^\infty$ be the set of eventually zero sequences in $\mathbb{R}$, that is $\mathbb{R}^\infty = \big\{x = (x_j)_{j\geq 1} \in \mathbb{R}^\omega\,\big|\,\exists\, m\in\mathbb{Z}^+\ \forall\, j\in\mathbb{Z}^+\,(j\geq m \Longrightarrow x_j=0)\big\}$. For $x, y \in \mathbb{R}^\infty$, define $x\bullet y = \sum_{n=1}^\infty x_n y_n$ and $|x| = \big(x\bullet x\big)^{1/2}$.

When $(x_n)_{n\geq 1}$ is a sequence in $\mathbb{R}^\infty$, each $x_n \in \mathbb{R}^\infty$, and we can write $x_n = (x_{n,j})_{j\geq 1} = (x_{n,1}, x_{n,2}, x_{n,3}, \cdots)$. For a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^\infty$ and an element $a \in \mathbb{R}^\infty$, we say the sequence $(x_n)_{n\geq 1}$ *converges* to a in $\mathbb{R}^\infty$, and we write $x_n \to a$ in $\mathbb{R}^\infty$ or $\lim\limits_{n\to\infty} x_n = a$ in $\mathbb{R}^\infty$, when $\forall\,\epsilon>0\ \exists\, m\in\mathbb{Z}^+\ \forall\, n\in\mathbb{Z}^+\ (n\geq m \Longrightarrow |x_n - a| < \epsilon)$, we say that $(x_n)_{n\geq 1}$ is *bounded* when $\exists\, r\geq 0\ \forall\, n\in\mathbb{Z}^+\ |x_n| \leq r$, and we say that $(x_n)_{n\geq 1}$ is *Cauchy* when $\forall\,\epsilon>0\ \exists\, m\in Z^+\ \forall\, k,\ell\in\mathbb{Z}^+\,(k,\ell\geq m \Longrightarrow |x_k - x_\ell| < \epsilon)$.

(a) Prove that for all sequences $(x_n)_{n\geq 1}$ in $\mathbb{R}^\infty$, and all $a \in \mathbb{R}^\infty$, if $\lim\limits_{n\to\infty} x_n = a$ in $\mathbb{R}^\infty$ then $\lim\limits_{n\to\infty} x_{n,j} = a_j$ for all $j \in \mathbb{Z}^+$, but that the converse does not hold.

Solution: Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}^\infty$ and let $a \in \mathbb{R}^\infty$. Suppose that $\lim\limits_{n\to\infty} x_n = a$ in $\mathbb{R}^\infty$. We claim that $\lim\limits_{n\to\infty} x_{n,j} = a_j$ for all $j \in \mathbb{Z}^+$. Let $j \in \mathbb{Z}^+$. Note that $|x_{n,j} - a_j|^2 \leq \sum_{i=1}^\infty (x_{n,i} - a_i)^2 = |x_n - a|^2$. Since $|x_{n,j} - a_j| \leq |x_n - a|$ and $\lim\limits_{n\to\infty} x_n = a$ in $\mathbb{R}^\infty$, it follows that $\lim\limits_{n\to\infty} x_{n,j} = a_k$ in $\mathbb{R}$: indeed given $\epsilon > 0$, we can choose $m \in \mathbb{Z}^+$ so that $n \geq m \Longrightarrow |x_n - a| < \epsilon$, and then, for $n \geq m$, we have $|x_{n,j} - a_j| \leq |x_n - a| < \epsilon$.

To see that the converse does not hold, for each $n \in \mathbb{Z}^+$, let $x_n = \frac{1}{\sqrt{n}}\sum_{k=1}^n e_k = \big(\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}}, 0, 0, \cdots\big)$, where $e_k = (0, 0, \cdots, 0, 1, 0, 0, \cdots)$ is the k^{th} standard basis vector for $\mathbb{R}^\infty$. For each index $j \in \mathbb{Z}^+$ we have $x_{n,j} = \frac{1}{\sqrt{n}}$ for all $n \geq j$ so that $\lim\limits_{n\to\infty} x_{n,j} = 0$ in $\mathbb{R}$. But for $a = 0 = (0, 0, 0, \cdots)$ we do not have $\lim\limits_{n\to\infty} x_n = a$ in $\mathbb{R}^\infty$ because $|x_n - 0| = |x_n| = 1$ for all $n \in \mathbb{Z}^+$.

(b) Prove that for all sequences $(x_n)_{n\geq 1}$ in $\mathbb{R}^\infty$, if the sequence $(x_n)_{n\geq 1}$ converges in $\mathbb{R}^\infty$ (to some $a \in \mathbb{R}^\infty$) then it is Cauchy, but that the converse does not hold.

Solution: Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}^\infty$. Suppose that $(x_n)_{n\geq 1}$ converges in $\mathbb{R}^\infty$ and let $a = \lim\limits_{n\to\infty} x_n$ in $\mathbb{R}^\infty$. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ so that $n \geq m \Longrightarrow |x_n - a| < \frac{\epsilon}{2}$. Then when $k, \ell \geq m$ we have $|x_k - x_\ell| = \big|(x_k - a) - (x_\ell - a)\big| \leq |x_k - a| + |x_\ell - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus $(x_n)_{n\geq 1}$ is Cauchy.

To see that the converse does not hold, for each $n \in \mathbb{Z}^+$ let $x_n = \sum_{k=1}^n \frac{1}{2^k} e_k = \big(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots, \frac{1}{2^n}, 0, 0, \cdots\big)$. We claim that $(x_n)_{n\geq 1}$ is Cauchy. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ so that $\frac{1}{2^m} < \epsilon$. Let $k, \ell \in \mathbb{Z}^+$ with $m \leq k < \ell$. Then we have $|x_k - x_\ell|^2 = \big|\sum_{j=k+1}^\ell \frac{1}{2^j} e_j\big|^2 = \sum_{j=k+1}^\ell \frac{1}{4^j} \leq \sum_{j=k+1}^\infty \frac{1}{4^j} = \frac{1}{4^k}$ so that $|x_k - x_\ell| \leq \frac{1}{2^k} \leq \frac{1}{2^m} < \epsilon$. Thus $(x_n)_{n\geq 1}$ is Cauchy, as claimed. Suppose, for a contradiction, that $(x_n)_{n\geq 1}$ converges in $\mathbb{R}^\infty$ and let $a = \lim\limits_{n\to\infty} x_n \in \mathbb{R}^\infty$. Note that for each $j \in \mathbb{Z}^+$, we have $x_{n,j} = \frac{1}{2^j}$ for all $n \geq j$ so that $\lim\limits_{n\to\infty} x_{n,j} = \frac{1}{2^j}$. By Part (a), for each $j \in \mathbb{Z}^+$ we must have $a_j = \lim\limits_{n\to\infty} x_{n,j} = \frac{1}{2^j}$ so that $a = \sum_{j=1}^\infty \frac{1}{2^j} e_j = \big(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\big)$. But then $a \notin \mathbb{R}^\infty$, which gives the desired contradiction.

(c) Determine whether every bounded sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^\infty$ has a convergent subsequence $(x_{n_k})_{k\geq 1}$ in $\mathbb{R}^\infty$.

Solution: This is not true. For example, consider the sequence $x_n = e_n = (0, \cdots, 0, 1, 0, \cdots)$ for $n \in \mathbb{Z}^+$. Note that $(x_n)_{n\geq 1}$ is bounded since $|x_n| = 1$ for all $n \in \mathbb{Z}^+$. Let $(x_{n_k})_{k\geq 1}$ be any subsequence. Note that for $k, \ell \in \mathbb{Z}^+$ with $k \neq \ell$ we have $|x_{n_k} - x_{n_\ell}| = |e_{n_k} - e_{n_\ell}| = \sqrt{2}$, and so the sequence $(x_{n_k})_{k\geq 1}$ is not Cauchy (if it was Cauchy, then we would be able to choose $k, \ell \in \mathbb{Z}^+$ with $k < \ell$ such that $|x_{n_k} - x_{n_\ell}| < \sqrt{2}$). Since $(x_{n_k})_{k\geq 1}$ is not Cauchy, it does not converge, by Part (b).