Lecture Notes for MATH 137 and MATH 138

Single Variable Calculus

by Stephen New

Chapter 1. Exponential and Trigonometric Functions

1.1 Definition: Let X and Y be sets and let $f : X \to Y$. We say that f is **injective** (or **one-to-one**, written as $1{:}1$) when for every $y \in Y$ there exists at most one $x \in X$ such that $f(x) = y$. Equivalently, f is injective when for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$. We say that f is **surjective** (or **onto**) when for every $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$. Equivalently, f is surjective when $\text{Range}(f) = Y$. We say that f is **bijective** (or **invertible**) when f is both injective and surjective, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f(x) = y$. When f is bijective, we define the **inverse** of f to be the function $f^{-1} : Y \to X$ such that for all $y \in Y$, $f^{-1}(y)$ is equal to the unique element $x \in X$ such that $f(x) = y$. Note that when f is bijective so is f^{-1}, and in this case we have $(f^{-1})^{-1} = f$.

1.2 Example: Let $f(x) = \frac{1}{3}\sqrt{12x - x^2}$ for $0 \le x \le 6$. Show that f is injective and find a formula for its inverse function.

Solution: Note that when $0 \le x \le 6$ (indeed when $0 \le x \le 12$) we have $12x - x^2 = x(12 - x) \ge 0$, so that $\frac{1}{3}\sqrt{12x - x^2}$ exists, and we have $12x - x^2 = 36 - (x - 6)^2 \le 36$ so that $\frac{1}{3}\sqrt{12x - x^2} \le \frac{1}{3}\sqrt{36} = 2$. Thus if $0 \le x \le 6$ then $f(x) = \frac{1}{3}\sqrt{12x - x^2}$ exists and we have $0 \le f(x) \le 2$. Let $x, y \in \mathbf{R}$ with $0 \le x \le 6$ and $0 \le y \le 2$. Then we have

$$\begin{aligned}
y = f(x) &\iff y = \tfrac{1}{3}\sqrt{12x - x^2} \\
&\iff 3y = \sqrt{12x - x^2} \\
&\iff 9y^2 = 12x - x^2 \quad \text{, since } y \ge 0 \\
&\iff x^2 - 12x + 9y^2 = 0 \\
&\iff x = \frac{12 \pm \sqrt{144 - 36y^2}}{2} = 6 \pm 3\sqrt{4 - y^2} \quad \text{, by the Quadratic Formula} \\
&\iff x = 6 - 3\sqrt{4 - y^2} \quad \text{since } x \le 6.
\end{aligned}$$

Verify that when $0 \le y \le 2$ we have $0 \le 4 - y^2 \le 4$ so that $\sqrt{4 - y^2}$ exists and we have $0 \le 6 - 3\sqrt{4 - y^2} \le 6$. Thus when we consider f as a function $f : [0, 6] \to [0, 2]$, it is bijective and its inverse $f^{-1} : [0, 2] \to [0, 6]$ is given by $f^{-1}(y) = 6 - 3\sqrt{4 - y^2}$.

1.3 Definition: Let $f : A \subseteq \mathbf{R} \to \mathbf{R}$. We say that f is **even** when $f(-x) = f(x)$ for all $x \in A$ and we say that f is **odd** when $f(-x) = -f(x)$ for all $x \in A$.

1.4 Definition: Let $f : A \subseteq \mathbf{R} \to \mathbf{R}$. We say that f is **increasing** (on A) when it has the property that for all $x, y \in A$, if $x < y$ then $f(x) < f(y)$, and we say f is **decreasing** (on A) when for all $x, y \in A$ with $x < y$ we have $f(x) > f(y)$. We say that f is **monotonic** when f is either increasing or decreasing. Note that every monotonic function is injective.

1.5 Remark: We assume familiarity with exponential, logarithmic, trigonometric and inverse trigonometric functions. These functions can be defined rigorously. We shall give a brief description of how one can define the exponential and logarithmic function rigorously, and we shall provide an informal (non-rigorous) description of the trigonometric and inverse trigonometric functions, and we shall summarize some of their properties (without giving rigorous proofs).

1.6 Definition: Let us outline one possible way to define the value of x^y for suitable real numbers $x, y \in \mathbf{R}$. First we define $x^0 = 1$ for all $x \in \mathbf{R}$. Then for $n \in \mathbf{Z}$ with $n \geq 1$ we define x^n recursively by $x^n = x \cdot x^{n-1}$ for all $x \in \mathbf{R}$. Also, for $n \in \mathbf{Z}$ with $n \geq 1$ we define $x^{-n} = \frac{1}{x^n}$ for all $x \neq 0$. At this stage we have defined x^y for $y \in \mathbf{Z}$.

When $0 < n \in \mathbf{Z}$ is odd, for all $x \in \mathbf{R}$ we define $x^{1/n} = y$ where y is the unique real number such that $y^n = x$ (to be rigorous, one must prove that this number y exists and is unique). When $0 < n \in \mathbf{Z}$ is even, for $x \geq 0$ we define $x^{1/n} = y$ where y is the unique nonnegative real number such that $y^n = x$ (again, to be rigorous a proof is required). Also, for $0 < n \in \mathbf{Z}$ we define $x^{-1/n} = \frac{1}{x^{1/n}}$, which is defined for $x \neq 0$ if n is odd, and is defined for $x > 0$ when n is even. When $n, m \in \mathbf{Z}$ with $n > 0$ and $m > 0$ and $\gcd(n, m) = 1$, we define $x^{n/m} = (x^n)^{1/m}$, which is defined for all $x \in \mathbf{R}$ when m is odd and for $x \geq 0$ when m is even, and we define $x^{-n/m} = \frac{1}{x^{n/m}}$, defined for $x \neq 0$ when m is odd and for $x > 0$ when m is even. At this stage, we have defined x^y for $y \in \mathbf{Q}$.

For $y \in \mathbf{R}$, when $x > 0$ and $y \in \mathbf{R}$, we define

$$x^y = \lim_{t \to y, t \in \mathbf{Q}} x^t$$

(to be rigorous, one needs to define this limit and prove that it exists and is unique). Finally, we define $1^y = 1$ for all $y \in \mathbf{R}$ and we define $0^y = 0$ for all $y > 0$.

1.7 Theorem: *(Properties of Exponentials) Let $a, b, x, y \in \mathbf{R}$ with $a, b > 0$. Then*
(1) $a^0 = 1$,
(2) $a^{x+y} = a^b\, a^c$,
(3) $a^{x-y} = a^x / a^y$,
(4) $(a^x)^y = a^{xy}$,
(5) $(ab)^x = a^x b^x$.

Proof: We omit the proof.

1.8 Theorem: *(Properties of Power Functions)*
(1) When $a > 0$, the function $f : [0, \infty) \to [0, \infty)$ given by $f(x) = x^a$ is increasing and bijective and its inverse function is given by $f^{-1}(x) = x^{1/a}$.
(2) When $a < 0$, the function $f : (0, \infty) \to (0, \infty)$ given by $f(x) = x^a$ is decreasing and bijective and its inverse is given by $f^{-1}(x) = a^{1/x}$.

Proof: We omit the proof.

1.9 Definition: A function of the form $f(x) = x^a$ is called a **power function**.

1.10 Theorem: *(Properties of Exponential Functions)*
(1) When $a > 1$ the function $f : \mathbf{R} \to (0, \infty)$ given by $f(x) = a^x$ is increasing and bijective.
(2) When $0 < a < 1$ the function $f : \mathbf{R} \to (0, \infty)$ given by $f(x) = a^x$ is decreasing and bijective.

Proof: We omit the proof.

1.11 Definition: For $a > 0$ with $a \neq 1$, the function $f : \mathbf{R} \to (0, \infty)$ given by $f(x) = a^x$ is called the base a **exponential function**, its inverse function $f^{-1} : (0, \infty) \to \mathbf{R}$ is called the base a **logarithmic function**, and we write $f^{-1}(x) = \log_a x$. By the definition of the inverse function, we have $\log_a(a^x) = x$ for all $x \in \mathbf{R}$ and $a^{\log_a y} = y$ for all $y > 0$, and for all $x, y \in \mathbf{R}$ with $y > 0$ we have $y = a^x \iff x = \log_a y$.

1.12 Theorem: *(Properties of Logarithms) Let* $a, b, x, y \in (0, \infty)$. *Then*

(1) $\log_a 1 = 0$,
(2) $\log_a(xy) = \log_a x + \log_a y$,
(3) $\log_a(x/y) = \log_a x - \log_a y$,
(4) $\log_a(x^y) = y \log_a x$, *and*
(5) $\log_b x = \log_a x / \log_a b$,
(6) if $a > 1$, *the function* $g : (0, \infty) \to \mathbf{R}$ *given by* $g(x) = \log_a x$ *is increasing and bijective.*

Proof: We leave it, as an exercise, to show that these properties follow from the properties of exponentials.

1.13 Definition: There is a number $e \in \mathbf{R}$ called the **natural base**, with $e \cong 2.71828$, which can be defined in such a way that the function $f(x) = e^x$ is equal to its own derivative. We define

$$e = \lim_{n \to \infty} \left(1 + \tfrac{1}{n}\right)^n$$

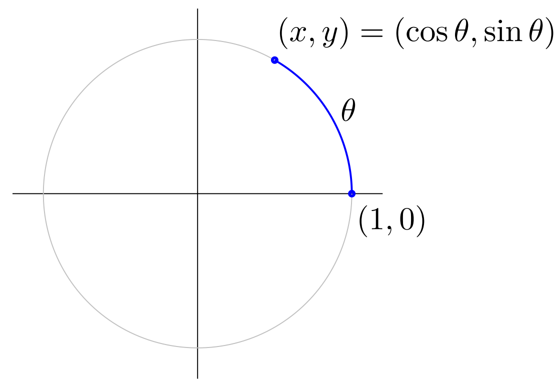
(to be rigorous, one must define this limit and prove that it exists and is unique). The logarithm to the base e is called the **natural logarithm**, and we write

$$\ln x = \log_e x \text{ for } x > 0.$$

1.14 Note: The properties of exponentials and logarithms in Theorems 1.7 and 1.12 give

$$e^0 = 1 \ , \ a^{x+y} = e^x e^y \ , \ e^{x-y} = e^x / e^y \ , \ (e^x)^y = e^{xy},$$
$$\ln 1 = 0 \ , \ \ln(xy) = \ln x + \ln y \ , \ \ln(x/y) = \ln x - \ln y \ , \ \ln x^y = y \ln x$$
$$\log_a x = \frac{\ln x}{\ln a} \quad \text{and} \quad a^x = e^{x \ln a}.$$

1.15 Definition: We define the trigonometric functions informally as follows. For $\theta \geq 0$, we define $\cos\theta$ and $\sin\theta$ to be the x- and y-coordinates of the point at which we arrive when we begin at the point $(1,0)$ and travel for a distance of θ units counterclockwise around the unit circle $x^2+y^2=1$. For $\theta \leq 0$, $\cos\theta$ and $\sin\theta$ are the x and y-coordinates of the point at which we arrive when we begin at $(1,0)$ and travel clockwise around the unit circle for a distance of $|\theta$ units. When $\cos\theta \neq 0$ we define $\sec\theta = 1/\cos\theta$ and $\tan\theta = \sin\theta/\cos\theta$, and when $\sin\theta \neq 0$ we define $\csc\theta = 1/\sin\theta$ and $\cot\theta = \cos\theta/\sin\theta$. (This definition is not rigorous because we did not define what it means to travel around the circle for a given distance).

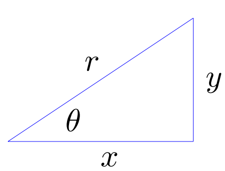


1.16 Definition: We define π, informally, to be the distance along the top half of the unit circle from $(1,0)$ to $(-1,0)$, and so we have $\cos\pi = -1$ and $\sin\pi = 0$. By symmetry, the distance from $(1,0)$ to $(0,1)$ along the circle is equal to $\frac{\pi}{2}$ so we also have $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

1.17 Theorem: *(Basic Trigonometric Properties) For $\theta \in \mathbf{R}$ we have*
(1) $\cos^2\theta + \sin^2\theta = 1$,
(2) $\cos(-\theta) = \cos\theta$ *and* $\sin(-\theta) = -\sin\theta$,
(3) $\cos(\theta+\pi) = -\cos\theta$ *and* $\sin(\theta+\pi) = -\sin\theta$,
(4) $\cos(\theta+2\pi) = \cos\theta$ *and* $\sin(\theta+2\pi) = \sin\theta$).

Proof: Informally, these properties can all be seen immediately from the above definitions. We omit a rigorous proof.

1.18 Theorem: *(Trigonometric Ratios) Let $\theta \in \left(0, \frac{\pi}{2}\right)$. For a right angle triangle with an angle of size θ and with sides of lengths x, y and r as shown, we have*



$$\cos\theta = \frac{x}{r}\ ,\ \sin\theta = \frac{y}{r}\ \text{ and }\ \tan\theta = \frac{y}{x}.$$

Proof: We can see this informally by scaling the picture in Definition 2.17 by a factor of r.

1.19 Theorem: *(Special Trigonometric Values) We have the following exact trigonometric values.*

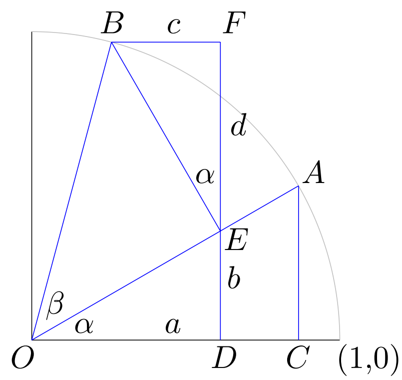
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Proof: This follows from the above theorem using certain particular right angled triangles.

1.20 Theorem: *(Trigonometric Sum Formulas) For $\alpha, \beta \in \mathbf{R}$ we have*

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \text{ , and}$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Proof: Informally, we can prove this with the help of a picture. The picture below illustrates the situation when $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$.



In the picture, O is the origin, A is the point with coordinates $(\cos\alpha, \sin\alpha)$ and B is the point $(x, y) = \big(\cos(\alpha+\beta), \sin(\alpha+\beta)\big)$. In triangle ODE we see that $\cos\alpha = \frac{OD}{OE} = \frac{a}{\cos\beta}$ and $\sin\alpha = \frac{DE}{OE} = \frac{b}{\cos\beta}$, and so $a = \cos\alpha\cos\beta$, $b = \sin\alpha\cos\beta$. In triangle EFB, verify that the angle at E has size α, and so we have $\cos\alpha = \frac{EF}{EB} = \frac{d}{\sin\beta}$ and $\sin\alpha = \frac{BF}{BE} = \frac{c}{\sin\beta}$, and so $c = \sin\alpha\sin\beta$, $d = \cos\alpha\sin\beta$. The x and y-coordinates of the point B are $x = a - c$ and $y = b + d$, and so

$$\cos(\alpha+\beta) = x = a - c = \cos\alpha\cos\beta - \sin\alpha\sin\beta \text{ , and}$$
$$\sin(\alpha+\beta) = y = b + d = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

This proves the theorem (informally) in the case that $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$. One can then show that the theorem holds for all $\alpha, \beta \in \mathbf{R}$ by using the Basic Trigonometric Properties (2), (3) and (4).

1.21 Theorem: *(Double Angle Formulas) For all $x, y \in \mathbf{R}$ we have*

(1) $\sin 2x = 2\sin x\cos x$ *and* $\cos 2x = \cos^2 - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$, *and*

(2) $\cos^2 x = \dfrac{1+\cos 2x}{2}$ *and* $\sin^2 x = \dfrac{1-\cos 2x}{2}$.

Proof: The proof is left as an exercise.

1.22 Theorem: *(Trigonometric Functions)*

(1) The function $f : [0, \pi] \to [-1, 1]$ defined by $f(x) = \cos x$ is decreasing and bijective.
(2) The function $g : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ given by $g(x) = \sin x$ is increasing and bijective.
(3) The function $h : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ given by $h(x) = \tan x$ is increasing and bijective.

Proof: We omit the proof.

1.23 Definition: The inverses of the functions f, g and h in the above theorem are called the **inverse cosine**, the **inverse sine**, and the **inverse tangent** functions. We write $f^{-1}(x) = \cos^{-1} x$, $g^{-1} = \sin^{-1} x$ and $h^{-1}(x) = \tan^{-1} x$. By the definition of the inverse function, we have

1.24 Definition: Let A and B be sets and let $c \in F$. Let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$. We define the functions cf , $f + g$, $f - g$, $f \cdot g$: $A \cap B \to \mathbf{R}$ by

$$\begin{aligned}(cf)(x) &= c\,f(x)\\ (f+g)(x) &= f(x) + g(x)\\ (f-g)(x) &= f(x) - g(x)\\ (f \cdot g)(x) &= f(x)g(x)\end{aligned}$$

for all $x \in A \cap B$, and for $C = \{x \in A \cap B \,|\, g(x) \neq 0\}$ we define $f/g : C \to \mathbf{R}$ by

$$(f/g)(x) = f(x)/g(x)$$

for all $x \in C$.

1.25 Definition: A **polynomial function** (over $\mathbf{R}$) is a function $f : \mathbf{R} \to \mathbf{R}$ which can be obtained from the functions 1 and x using (finitely many applications of) the operations cf, $f + g$, $f - g$, $f \cdot g$ and $f \circ g$. In other words, a polynomial is a function of the form

$$f(x) = \sum_{i=0}^{n} c_i x^i = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

for some $n \in \mathbf{N}$ and some $c_i \in F$. The numbers c_i are called the **coefficients** of the polynomial and when $c_n \neq 0$ the number n is called the **degree** of the polynomial.

1.26 Definition: A **rational function** (over $\mathbf{R}$) is a function $f : A \subseteq \mathbf{R} \to \mathbf{R}$ which can be obtained from the functions 1 and x using (finitely many applications of) the operations cf, $f + g$, $f - g$, $f \cdot g$, f/g and $f \circ g$. In other words, a rational function is a function of the form

$$f(x) = p(x)/q(x)$$

for some polynomials p and q.

1.27 Definition: The functions 1, x, $x^{1/n}$ with $0 < n \in \mathbf{Z}$, e^x, $\ln x$, $\sin x$ and $\sin^{-1} x$, are called the **basic elementary functions**. An **elementary function** is any function $f : A \subseteq \mathbf{R} \to \mathbf{R}$ which can be obtained from the basic elementary functions using (finitely many applications of) the operations cf, $f + g$, $f - g$, $f \cdot g$, f/g and $f \circ g$.

1.28 Example: The following functions are elementary

$$\begin{aligned}|x| &= \sqrt{x^2},\\ \cos x &= \sin\left(x + \tfrac{\pi}{2}\right),\\ \tan^{-1} x &= \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right),\\ f(x) &= \frac{e^{\sqrt{x}+\sin x}}{\tan^{-1}(\ln x)}\end{aligned}$$

We shall see later that every elementary function is continuous in its domain, so any function which is discontinuos at a point in its domain cannot be elementary.

Chapter 2. Limits of Sequences

2.1 Notation: We write $\mathbf{N} = \{0, 1, 2, \cdots\}$ for the set of **natural numbers** (which we take to include the number 0), $\mathbf{Z}^+ = \{1, 2, 3, \cdots\}$ for the set of **positive integers**, $\mathbf{Z} = \{0, \pm 1, \pm 2, \cdots\}$ for the set of all **integers**, $\mathbf{Q}$ for the set of **rational numbers** and we write $\mathbf{R}$ for the set of **real numbers**. We assume familiarity with the sets $\mathbf{N}$, $\mathbf{Z}^+$, $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ and with the algebraic operations $+$, $-$, $\times$, $\div$ and the order relations $<$, $\leq$, $>$, $\geq$ on these sets.

2.2 Definition: For $p \in \mathbf{Z}$, let $\mathbf{Z}_{\geq p} = \{k \in \mathbf{Z} | k \geq p\}$. A **sequence** in a set A is a function of the form $x : \mathbf{Z}_{\geq p} \to A$ for some $p \in \mathbf{Z}$. Given a sequence $x : \mathbf{Z}_{\geq p} \to A$, the k^{th} **term** of the sequence is the element $x_k = x(k) \in A$, and we denote the sequence x by

$$(x_k)_{k \geq p} = (x_k | k \geq p) = (x_p, x_{p+1}, x_{p+2}, \cdots).$$

Note that the range of the sequence $(x_k)_{k \geq p}$ is the set $\{x_k\}_{k \geq p} = \{x_k | k \geq p\}$.

2.3 Definition: Let $(x_k)_{k \geq p}$ be a sequence in $\mathbf{R}$. For $a \in \mathbf{R}$ we say that the sequence $(x_k)_{k \geq p}$ **converges** to a (or that the **limit** of $(x_k)_{k \geq p}$ is equal to a), and we write $x_k \to a$ (as $k \to \infty$), or we write $\lim\limits_{k \to \infty} x_k = a$, when

$$\forall\, \epsilon > 0\ \exists\, m \in \mathbf{Z}_{\geq p}\ \forall\, k \in \mathbf{Z}_{\geq p}\ \big(k \geq m \Longrightarrow |x_k - a| < \epsilon\big).$$

We say that the sequence $(x_k)_{k \geq p}$ **converges** (in $\mathbf{R}$) when there exists $a \in \mathbf{R}$ such that $(x_k)_{k \geq p}$ converges to a. We say that the sequence $(x_k)_{k \geq p}$ **diverges** (in $\mathbf{R}$) when it does not converge (to any $a \in \mathbf{R}$). We say that $(x_k)_{k \geq p}$ **diverges to infinity**, or that the limit of $(x_k)_{k \geq p}$ is equal to **infinity**, and we write $x_k \to \infty$ (as $k \to \infty$), or we write $\lim\limits_{k \to \infty} x_k = \infty$, when

$$\forall\, r \in \mathbf{R}\ \exists\, m \in \mathbf{Z}_{\geq p}\ \forall\, k \in \mathbf{Z}_{\geq p}\ \big(k \geq m \Longrightarrow x_k > r\big).$$

Similarly we say that $(x_k)_{k \geq p}$ **diverges to** $-\infty$, or that the limit of $(x_k)_{k \geq p}$ is equal to **negative infinity**, and we write $x_k \to -\infty$ (as $k \to \infty$), or we write $\lim\limits_{k \to \infty} x_k = -\infty$ when

$$\forall\, r \in \mathbf{R}\ \exists\, m \in \mathbf{Z}_{\geq p}\ \forall\, k \in \mathbf{Z}_{\geq p}\ \big(k \geq m \Longrightarrow x_k < r\big).$$

2.4 Example: Let $(x_k)_{k \geq 0}$ be the sequence in $\mathbf{R}$ given by $x_k = \frac{(-2)^k}{k!}$ for $k \geq 0$. Show that $\lim\limits_{k \to \infty} x_k = 0$.

Solution: Note that for $k \geq 2$ we have $|x_k| = \frac{2^k}{k!} = \left(\frac{2}{1}\right)\left(\frac{2}{2}\right)\left(\frac{2}{3}\right) \cdots \left(\frac{2}{k-1}\right)\left(\frac{2}{k}\right) \leq \frac{2}{1} \cdot \frac{2}{n} = \frac{4}{n}$. Given $\epsilon \in \mathbf{R}$ with $\epsilon > 0$, we can choose $m \in \mathbf{Z}_{\geq 2}$ with $m > \frac{4}{\epsilon}$ (by the Archimedean Property of $\mathbf{Z}$ in $\mathbf{R}$), and then for all $k \geq m$ we have $|x_k - 0| = |x_k| \leq \frac{4}{k} \leq \frac{4}{m} < \epsilon$. Thus $\lim\limits_{k \to \infty} x_k = 0$, by the definition of the limit.

2.5 Example: Let $(a_k)_{k \geq 0}$ be the **Fibonacci sequence** in $\mathbf{R}$, which is defined recursively by $a_0 = 0$, $a_1 = 1$ and by $a_k = a_{k-1} + a_{k-2}$ for $k \geq 2$. Show that $\lim\limits_{k \to \infty} a_k = \infty$.

Solution: We have $a_0 = 0$, $a_1 = 1$, $a_2 = 1$ and $a_3 = 2$. Note that $a_k \geq k - 1$ when $k \in \{0, 1, 2, 3\}$. Let $n \geq 4$ and suppose, inductively, that $a_k \geq k - 1$ for all $k \in \mathbf{Z}$ with $0 \leq k < n$. Then $a_n = a_{n-1} + a_{n-2} \geq (n-2) + (n-3) = n + n - 5 \geq n + 4 - 5 = n - 1$. By the Strong Principle of Induction, we have $a_n \geq n - 1$ for all $n \geq 0$. Given $r \in \mathbf{R}$ we can choose $m \in \mathbf{Z}_{\geq 0}$ with $m > r + 1$, and then for all $k \geq m$ we have $a_k \geq k - 1 \geq m - 1 > r$. Thus $\lim\limits_{k \to \infty} a_k = \infty$ by the definition of the limit.

2.6 Example: Let $x_k = (-1)^k$ for $k \geq 0$. Show that $(x_k)_{k\geq 0}$ diverges.

Solution: Suppose, for a contradiction, that $(x_k)_{k\geq 0}$ converges and let $a = \lim_{k\to\infty} x_k$. By taking $\epsilon = 1$ in the definition of the limit, we can choose $m \in \mathbf{Z}$ so the for all $k \in \mathbf{N}$, if $k \geq m$ then $|x_k - a| < 1$. Choose $k \in \mathbf{N}$ with $2k \geq m$. Since $|x_{2k} - a| < 1$ and $x_{2k} = (-1)^{2k} = 1$, we have $|1 - a| < 1$ so that $0 < a < 2$. Since $|x_{2k+1} - a| < 1$ and $x_{2k+1} = (-1)^{2k+1} = -1$, we also have $|-1-a| < 1$ which implies that $-2 < a < 0$. But then we have $a < 0$ and $a > 0$, which is not possible.

2.7 Theorem: *(Independence of the Limit on the Initial Terms) Let $(x_k)_{k\geq p}$ be a sequence in* $\mathbf{R}$.

(1) If $q \geq p$ and $y_k = x_k$ for all $k \geq q$, then $(x_k)_{k\geq p}$ converges if and only if $(y_k)_{k\geq q}$ converges, and in this case $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k$.

(2) If $l \geq 0$ and $y_k = x_{k+l}$ for all $k \geq p$, then $(x_k)_{k\geq p}$ converges if and only if $(y_k)_{k\geq p}$ converges, and in this case $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k$.

Proof: We prove Part 1 and leave the proof of Part 2 as an exercise. Let $q \geq p$ and let $y_k = x_k$ for $k \geq q$. Suppose $(x_k)_{k\geq p}$ converges and let $a = \lim_{k\to\infty} x_k$. Let $\epsilon > 0$. Choose $m \in \mathbf{Z}$ so that for all $k \in \mathbf{Z}_{\geq p}$, if $k \geq m$ then $|x_k - a| < \epsilon$. Let $k \in \mathbf{Z}_{\geq q}$ with $k \geq m$. Since $q \geq p$ we also have $k \in \mathbf{Z}_{\geq p}$ and so $|y_k - a| = |x_k - a| < \epsilon$. Thus $(y_k)_{k\geq q}$ converges with $\lim_{k\to\infty} y_k = a$. Conversely, suppose that $(y_k)_{k\geq q}$ converges and let $a = \lim_{k\to\infty} y_k$. Let $\epsilon > 0$. Choose $m_1 \in \mathbf{Z}$ so that for all $k \in \mathbf{Z}_{\geq q}$, if $k \geq m_1$ then $|y_k - a| < \epsilon$. Choose $m = \max\{m_1, q\}$. Let $k \in \mathbf{Z}_{\geq p}$ with $k \geq m$. Since $k \geq m$, we have $k \geq q$ and $k \geq m_1$ and so $|x_k - a| = |y_k - a| < \epsilon$. Thus $(x_k)_{k\geq p}$ converges with $\lim_{k\to\infty} x_k = a$.

2.8 Remark: Because of the above theorem, we often denote the sequence $(x_k)_{k\geq p}$ simply as (x_k), omitting the initial index p from our notation. Also, in the statements of some theorems and in some proofs we select a particular starting point, often $p = 1$, with the understanding that any other starting value would work just as well.

2.9 Theorem: *(Uniqueness of the Limit) Let (x_k) be a sequence in $\mathbf{R}$. If (x_k) has a limit (finite or infinite) then the limit is unique.*

Proof: Suppose, for a contradiction, that $x_k \to \infty$ and $x_k \to -\infty$. Since $x_k \to \infty$ we can choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \Longrightarrow x_k > 0$. Since $x_k \to -\infty$ we can choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow x_k < 0$. Choose any $k \in \mathbf{Z}_{\geq p}$ with $k \geq m_1$ and $k \geq m_2$. Then $x_k > 0$ and $x_k < 0$, which is not possible.

Suppose, for a contradiction, that $x_k \to \infty$ and $x_k \to a \in F$. Since $x_k \to a$ we can choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \Longrightarrow |x_k - a| < 1$. Since $x_k \to \infty$ we can choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow x_k > a + 1$. Choose any $k \in \mathbf{Z}_{\geq p}$ with $k \geq m_1$ and $k \geq m_2$. Then we have $|x_k - a| < 1$ so that $x < a + 1$ and we have $x_k > a + 1$, which is not possible. Similarly, it is not possible to have $x_k \to -\infty$ and $x_k \to a \in F$.

Finally suppose, for a contradiction, that $x_k \to a$ and $x_k \to b$ where $a, b \in F$ with $a \neq b$. Since $x_k \to a$ we can choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \Longrightarrow |x_k - a| < \frac{|a-b|}{2}$. Since $x_k \to b$ we can choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow |x_k - b| < \frac{|a-b|}{2}$. Choose any $k \in \mathbf{Z}_{\geq p}$ with $k \geq m_1$ and $k \geq m_2$. Then we have $|x_k - a| < \frac{b-a}{2}$ and $|x_k - b| < \frac{b-a}{2}$ and so, using the Triangle Inequality, we have

$$|a - b| = |a - x_k + x_k - b| \leq |x_k - a| + |x_k - b| < \tfrac{|a-b|}{2} + \tfrac{|a-b|}{2} = |a - b|,$$

which is not possible.

2.10 Theorem: *(Basic Limits) For $a \in \mathbf{R}$ we have*

$$\lim_{k\to\infty} a = a \ , \ \lim_{k\to\infty} k = \infty \ \ \text{and} \ \ \lim_{k\to\infty} \frac{1}{k} = 0.$$

Proof: The proof is left as an exercise.

2.11 Theorem: *(Operations on Limits) Let (x_k) and (y_k) be sequences in $\mathbf{R}$ and let $c \in \mathbf{R}$. Suppose that (x_k) and (y_k) both converge with $x_k \to a$ and $y_k \to b$. Then*
(1) $(c\,x_k)$ converges with $c\,x_k \to ca$,
(2) $(x_k + y_k)$ converges with $(x_k + y_k) \to a + b$,
(3) $(x_k - y_k)$ converges with $(x_k - y_k) \to a - b$,
(4) $(x_k y_k)$ converges with $x_k y_k \to ab$, and
(5) if $b \neq 0$ then (x_k/y_k) converges with $x_k/y_k \to a/b$.

Proof: We prove Parts 4 and 5 leaving the proofs of the other parts as an exercise. First we prove Part 4. Note that for all k we have

$$|x_k y_k - ab| = |x_k y_k - x_k b + x_k b - ab| \leq |x_k y_k - x_k b| + |x_k b - ab| = |x_k|\,|y_k - b| + |b||x_k - a|.$$

Since $x_k \to a$ we can choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \Longrightarrow |x_k - a| < 1$ and we can choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow |x_k - a| < \frac{\epsilon}{2(1+|b|)}$. Since $y_k \to b$ we can choose $m_3 \in \mathbf{Z}$ so that $k \geq m_3 \Longrightarrow |y_k - b| < \frac{\epsilon}{2(1+|a|)}$. Let $m = \max\{m_1, m_2, m_3\}$ and let $k \geq m$. Then we have $|x_k - a| < 1$, $|x_k - a| < \frac{\epsilon}{2(1+|b|)}$ and $|x_k - b| < \frac{\epsilon}{2(1+|a|)}$. Since $|x_k - a| < 1$, we have $|x_k| = |x_k - a + a| \leq |x_k - a| + |a| < 1 + |a|$. By our above calculation (where we found a bound for $|x_k y_k - ab|$) we have

$$\begin{aligned} |x_k y_k - ab| &\leq |x_k||y_k - b| + |b||x_k - a| \leq (1+|a|)|y_k - b| + (1+|b|)|x_k - a| \\ &< (1+|a|)\tfrac{\epsilon}{2(1+|a|)} + (1+|b|)\tfrac{\epsilon}{2(1+|b|)} = \epsilon. \end{aligned}$$

Thus we have $x_k y_k \to ab$, by the definition of the limit.

To prove Part 5, suppose that $b \neq 0$. Since $y_k \to b \neq 0$, we can choose $m_1 \in \mathbf{Z}$ so that that $k \geq m_1 \Longrightarrow |y_k - b| < \frac{|b|}{2}$. Then for $k \geq m_1$ we have

$$|b| = |b - y_k + y_k| \leq |b - y_k| + |y_k| < \tfrac{|b|}{2} + |y_k|$$

so that

$$|y_k| > |b| - \tfrac{|b|}{2} = \tfrac{|b|}{2} > 0.$$

In particular, we remark that when $k \geq m_1$ we have $y_k \neq 0$ so that $\frac{1}{y_k}$ is defined. Note that for all $k \geq m_1$ we have

$$\left| \frac{1}{y_k} - \frac{1}{b} \right| = \frac{|b - y_k|}{|y_k|\,|b|} \leq \frac{|b - y_k|}{\frac{|b|}{2} \cdot |b|} = \frac{2}{|b|^2} \cdot |y_k - b|.$$

Let $\epsilon > 0$. Choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \Longrightarrow |y_k - b| < \frac{|b|^2 \epsilon}{2}$. Let $m = \max\{m_1, m_2\}$. For $k \geq m$ we have $k \geq m_1$ and $k \geq m_2$ and so $|y_k| > \frac{|b|^2}{2}$ and $|y_k - b| < \frac{|b|^2\epsilon}{2}$ and so

$$\left| \frac{1}{y_k} - \frac{1}{b} \right| \leq \tfrac{2}{|b|^2} \cdot |y_k - b| < \tfrac{2}{|b|^2} \cdot \tfrac{|b|^2 \epsilon}{2} = \epsilon.$$

This proves that $\lim\limits_{k\to\infty} \frac{1}{y_k} = \frac{1}{b}$. Using Part 4, we have $\lim\limits_{k\to\infty} \frac{x_k}{y_k} = \lim\limits_{k\to\infty} \left(x_k \cdot \frac{1}{y_k} \right) = a \cdot \frac{1}{b} = \frac{a}{b}$.

2.12 Example: Let $x_k = \frac{k^2+1}{2k^2+k+3}$ for $k \geq 0$. Find $\lim\limits_{k\to\infty} x_k$.

Solution: We have $x_k = \frac{k^2+1}{2k^2+k+2} = \frac{1+\left(\frac{1}{k}\right)^2}{2+\frac{1}{k}+3\cdot\left(\frac{1}{k}\right)^2} \longrightarrow \frac{1+0^2}{2+0+3\cdot 0^2} = \frac{1}{2}$ where we used the Basic Limits $1 \to 1$, $2 \to 2$ and $\frac{1}{k} \to 0$ together with Operations on Limits.

2.13 Definition: The above theorem can be extended to include many situations involving infinite limits. To deal with these cases, we define the set of **extended real numbers** to be the set

$$\widehat{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}.$$

We extend the order relation $<$ on $\mathbf{R}$ to an order relation on $\widehat{\mathbf{R}}$ by defining $-\infty < \infty$ and $-\infty < a$ and $a < \infty$ for all $a \in \mathbf{R}$. We partially extend the operations $+$ and $\times$ to $\widehat{\mathbf{R}}$ as follows: for $a \in \mathbf{R}$ we define

$$\infty + \infty = \infty\ ,\ \infty + a = \infty\ ,\ (-\infty) + (-\infty) = -\infty\ ,\ (-\infty) + a\ ,$$
$$\infty \cdot \infty = \infty\ , (\infty)(-\infty) = -\infty\ ,\ (-\infty)(-\infty) = \infty\ ,$$
$$\infty \cdot a = \begin{cases} \infty \text{ if } a > 0 \\ -\infty \text{ if } a < 0 \end{cases} \text{ and } (-\infty)(a) = \begin{cases} -\infty \text{ if } a > 0, \\ \infty \text{ if } a < 0, \end{cases}$$

but other values, including $\infty + (-\infty)$, $\infty \cdot 0$ and $-\infty \cdot 0$ are left undefined in $\widehat{\mathbf{R}}$. In a similar way, we partially extend the inverse operations $-$ and $\div$ to $\widehat{\mathbf{R}}$. For example, for $a \in \mathbf{R}$ we define

$$\infty-(-\infty) = \infty\, , -\infty-\infty = -\infty\, , \infty-a = \infty\, , -\infty-a = -\infty\, , a-\infty = -\infty\, , a-(-\infty) = \infty\, ,$$
$$\frac{a}{\infty} = 0\ ,\ \frac{\infty}{a} = \begin{cases} \infty \text{ if } a > 0 \\ -\infty \text{ if } a < 0 \end{cases} \text{ and } \frac{-\infty}{a} = \begin{cases} -\infty \text{ if } a > 0 \\ \infty \text{ if } a < 0 \end{cases}$$

with other values, including $\infty - \infty$, $\frac{\infty}{\infty}$ and $\frac{\infty}{0}$, left undefined. The expressions which are left undefined in $\widehat{\mathbf{R}}$, including

$$\infty - \infty\ ,\ \infty \cdot 0\ ,\ \frac{\infty}{\infty}\ ,\ \frac{\infty}{0}\ ,\ \frac{a}{0}\, ,$$

are known as **indeterminate forms**.

2.14 Theorem: *(Extended Operations on Limits) Let (x_k) and (y_k) be sequences in $\mathbf{R}$. Suppose that $\lim\limits_{k\to\infty} x_k = u$ and $\lim\limits_{k\to\infty} y_k = v$ where $u, v \in \widehat{\mathbf{R}}$.*

(1) if $u + v$ is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{k\to\infty} (x_k + y_k) = u + v$,

(2) if $u - v$ is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{k\to\infty} (x_k - y_k) = u - v$,

(3) if $u \cdot v$ is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{k\to\infty} (x_k \cdot y_k) = u \cdot v$, and

(4) if u/v is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{k\to\infty} (x_k / y_k) = u/v$.

Proof: The proof is left as an exercise.

2.15 Theorem: *(Comparison) Let (x_k) and (y_k) be sequences in $\mathbf{R}$. Suppose that $x_k \leq y_k$ for all k. Then*

(1) if $x_k \to a$ and $y_k \to b$ then $a \leq b$,
(2) if $x_k \to \infty$ then $y_k \to \infty$, and
(3) if $y_k \to -\infty$ then $x_k \to -\infty$.

Proof: We prove Part 1. Suppose that $x_k \to a$ and $y_k \to b$. Suppose, for a contradiction, that $a > b$. Choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \implies |x_k - a| < \frac{a-b}{2}$. Choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \implies |y_k - b| < \frac{a-b}{2}$. Let $k = \max\{m_1, m_2\}$. Since $|x_k - a| < \frac{a-b}{2}$, we have $x_k > a - \frac{a-b}{2} = \frac{a+b}{2}$. Since $|y_k - b| < \frac{a-b}{2}$, we have $y_k < b + \frac{a-b}{2} = \frac{a+b}{2}$. This is not possible since $x_k \leq y_k$.

2.16 Example: Let $x_k = (\frac{3}{2} + \sin k)\ln k$ for $k \geq 1$. Find $\lim\limits_{k\to\infty} x_k$.

Solution: For all $k \geq 1$ we have $\sin k \geq -1$ so $(\frac{3}{2} + \sin k) \geq \frac{1}{2}$ and hence $x_k \geq \frac{1}{2}\ln k$. Since $x_k \geq \frac{1}{2}\ln k$ for all $k \geq 1$ and $\frac{1}{2}\ln k \longrightarrow \frac{1}{2} \cdot \infty = \infty$, it follows that $x_k \to \infty$ by the Comparison Theorem.

2.17 Theorem: *(Squeeze) Let (x_k), (y_k) and (z_k) be sequences in $\mathbf{R}$ and let $a \in \mathbf{R}$.*

(1) If $x_k \leq y_k \leq z_k$ for all k and $x_k \to a$ and $z_k \to a$ then $y_k \to a$.
(2) If $|x_k| \leq y_k$ for all k and $y_k \to 0$ then $x_k \to 0$.

Proof: We prove Part 1. Suppose that $x_k \leq y_k \leq z_k$ for all k, and suppose that $x_k \to a$ and $z_k \to a$. Let $\epsilon > 0$. Choose $m_1 \in \mathbf{Z}$ so that $k \geq m_1 \implies |x_k - a| < \epsilon$, choose $m_2 \in \mathbf{Z}$ so that $k \geq m_2 \implies |z_k - a| < \epsilon$ and let $m = \max\{m_1, m_2\}$. Then for $k \geq m$ we have $a - \epsilon < x_k \leq y_k \leq z_k < a + \epsilon$ and so $|y_k - a| < \epsilon$. Thus $y_k \to a$, as required.

2.18 Example: Let $x_k = \frac{k + \tan^{-1} k}{2k + \sin k}$ for $k \geq 1$. Find $\lim\limits_{k\to\infty} x_k$.

Solution: For all $k \geq 1$ we have $-\frac{\pi}{2} < \tan^{-1} k < \frac{\pi}{2}$ and $-1 \leq \sin k \leq 1$ and so

$$\frac{k - \frac{\pi}{2}}{2k+1} \leq \frac{k + \tan^{-1} k}{2k + \sin k} \leq \frac{k + \frac{\pi}{2}}{2k-1}.$$

As in previous examples, we have $\frac{k-\frac{\pi}{2}}{2k+1} \to \frac{1}{2}$ and $\frac{k+\frac{\pi}{2}}{2k-1} \to \frac{1}{2}$, and so $x_k = \frac{k+\tan^{-1} k}{2k+\sin k} \to \frac{1}{2}$ by the Squeeze Theorem.

2.19 Definition: Let (x_k) be a sequence in $\mathbf{R}$. For $a, b \in \mathbf{R}$, we say that the sequence (x_k) is **bounded above** by b when $x_k \leq b$ for all k, and we say that the sequence (x_k) is **bounded below** by a when $a \leq x_k$ for all k. We say (x_k) is **bounded above** when it is bounded above by some element $b \in \mathbf{R}$, we say that (x_k) is **bounded below** when it is bounded below by some $a \in \mathbf{R}$, and we say that (x_k) is **bounded** when it is bounded above and bounded below.

2.20 Definition: Let (x_k) be a sequence in $\mathbf{R}$. We say that (x_k) is **increasing** (for $k \geq p$) when for all $k, l \in \mathbf{Z}_{\geq p}$, if $k \leq l$ then $x_k \leq x_l$. We say that (x_k) is **strictly increasing** (for $k \geq p$) when for all $k, l \in \mathbf{Z}_{\geq p}$, if $k < l$ then $x_k < x_l$. Similarly, we say that (x_k) is **decreasing** when for all $k, l \in \mathbf{Z}_{\geq p}$, if $k \leq l$ the $x_k \geq x_l$ and we say that (x_k) is **strictly decreasing** when for all $k, l \in \mathbf{Z}_{\geq p}$, if $k < l$ the $x_k > x_l$. We say that (x_k) is **monotonic** when it is either increasing or decreasing.

2.21 Theorem: *(Monotonic Convergence) Let (x_k) be a sequence in $\mathbf{R}$.*

(1) Suppose (x_k) is increasing. If (x_k) is bounded above then it converges, and if (x_k) is not bounded above then $x_k \to \infty$.
(2) Suppose (x_k) is decreasing. If (x_k) is bounded below then it converges, and if (x_k) is not bounded below then $x_k \to -\infty$.

Proof: The statement of this theorem is intuitively reasonable, but it is quite difficult to prove. In most calculus courses this theorem is accepted axiomatically, without proof. A rigorous proof is often provided in analysis courses.

2.22 Example: Let $x_1 = \frac{4}{3}$ and let $x_{k+1} = 5 - \frac{4}{x_n}$ for $k \geq 1$. Determine whether (x_k) converges, and if so then find the limit.

Solution: Suppose, for now, that (x_k) does converge, say $x_k \to a$. By Independence of Converge on Initial Terms, we also have $x_{k+1} \to a$. Using Operations on Limits, we have $a = \lim\limits_{k\to\infty} x_{k+1} = \lim\limits_{k\to\infty} \left(5 - \frac{4}{x_k}\right) = 5 - \frac{4}{a}$. Since $a = 5 - \frac{4}{a}$, we have $a^2 = 5a - 4$ or equivalently $(a-1)(a-4) = 0$. We have proven that if the sequence converges then its limit must be equal to 1 or 4.

The first few terms of the sequence are $x_1 = \frac{4}{3}$, $x_2 = 2$ and $x_3 = 3$. Since the terms appear to be increasing, we shall try to prove that $1 \leq x_n \leq x_{n+1} \leq 4$ for all $n \geq 1$. This is true when $n = 1$. Suppose it is true when $n = k$. Then we have

$$1 \leq x_k \leq x_{k+1} \leq 4 \Longrightarrow 1 \geq \tfrac{1}{x_k} \geq \tfrac{1}{x_{k+1}} \geq \tfrac{1}{4} \Longrightarrow -4 \leq -\tfrac{4}{x_k} \leq -\tfrac{4}{x_{k+1}} \leq -1$$
$$\Longrightarrow 1 \leq 5 - \tfrac{4}{x_k} \leq 5 - \tfrac{4}{x_{k+1}} \leq 4 \Longrightarrow 1 \leq x_{k+1} \leq x_{k+2} \leq 4.$$

Thus, by the Principle of Induction, we have $1 \leq x_n \leq x_{n+1} \leq 4$ for all $n \geq 1$.

Since $x_n \leq x_{n+1}$ for all $n \geq 1$, the sequence is increasing, and since $x_n \leq 4$ for all $n \geq 1$, the sequence is bounded above by 4. By the Monotone Convergence Theorem, the sequence does converge. By the first paragraph, we know the limit must be either 1 or 4, and since the sequence starts at $x_1 = 2$ and increases, the limit must be 4.

Chapter 3. Limits of Functions and Continuity

3.1 Definition: Let $A \subseteq \mathbf{R}$ and let $a \in \mathbf{R}$. We say that a is a **limit point** of A when

$$\forall\, \delta > 0\ \exists\, x \in A\ \ 0 < |x - a| < \delta.$$

We say that a is a **limit point of A from below** (or **from the left**) when

$$\forall \delta > 0\ \exists x \in A \quad a - \delta < x < a\,.$$

We say that a is a **limit point of A from above** (or **from the right**) when

$$\forall \delta > 0\ \exists x \in A \quad a < x < a + \delta\,.$$

We say that A is **not bounded above** when $\forall m \in \mathbf{R}\ \exists x \in A\ \ x \geq m$, and we say that A is **not bounded below** when $\forall m \in \mathbf{R}\ \exists x \in A\ \ x \leq m$.

3.2 Example: Let A be a finite union of non-degenerate intervals in $\mathbf{R}$ (a non-degenerate interval is an interval which contains more than one point). The limit points of A are the points $a \in \mathbf{R}$ such that either $a \in A$ or a is an endpoint of one of the intervals. The limit points of A from below are the points $a \in \mathbf{R}$ such that either $a \in A$ or a is the right endpoint of one of the intervals. The set A is not bounded above when one of the intervals is of one of the forms (a, ∞), $[a, \infty)$ or $(-\infty, \infty) = \mathbf{R}$.

3.3 Definition: Let $A \subseteq \mathbf{R}$ and let $f : A \to \mathbf{R}$. When $a \in \mathbf{R}$ is a limit point of A, we make the following definitions.

(1) For $b \in \mathbf{R}$, we say that the **limit** of $f(x)$ as x tends to a is equal to b, and we write $\lim\limits_{x \to a} f(x) = b$ or we write $f(x) \to b$ as $x \to a$, when

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A\ \big(0 < |x - a| < \delta \Longrightarrow |f(x) - b| < \epsilon\big).$$

(2) We say the limit of $f(x)$ as x tends to a is equal to **infinity**, and we write $\lim\limits_{x \to a} f(x) = \infty$, or we write $f(x) \to \infty$ as $x \to a$, when

$$\forall r \in \mathbf{R}\ \exists \delta > 0\ \forall x \in A\ \big(0 < |x - a| < \delta \Longrightarrow f(x) > r\big).$$

(3) We say that the limit of $f(x)$ as x tends to a is equal to **negative infinity**, and we write $\lim\limits_{x \to a} f(x) = -\infty$, or we write $f(x) \to -\infty$ as $x \to a$, when

$$\forall r \in \mathbf{R}\ \exists \delta > 0\ \forall x \in A\ \big(0 < |x - a| < \delta \Longrightarrow f(x) < r\big).$$

When a is a limit point of A from below and $b \in \mathbf{R}$, we make the following definitions.

(4) $\lim\limits_{x \to a^-} f(x) = b \iff \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A\ \big(a - \delta < x < a \Longrightarrow |f(x) - b| < \epsilon\big)$.

(5) $\lim\limits_{x \to a^-} f(x) = \infty \iff \forall r \in \mathbf{R}\ \exists \delta > 0\ \forall x \in A\ \big(a - \delta < x < a \Longrightarrow f(x) > r\big)$.

(6) $\lim\limits_{x \to a^-} f(x) = -\infty \iff \forall r \in \mathbf{R}\ \exists \delta > 0\ \forall x \in A\ \big(a - \delta < x < a \Longrightarrow f(x) < r\big)$.

When a is a limit point of A from above and $b \in \mathbf{R}$, we make the following definitions.

(7) $\lim\limits_{x \to a^+} f(x) = b \iff \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A\ \big(a < x < a + \delta \Longrightarrow |f(x) - b| < \epsilon\big)$.

(8) $\lim\limits_{x \to a^+} f(x) = \infty \iff \forall r \in \mathbf{R}\ \exists \delta > 0\ \forall x \in A\ \big(a < x < a + \delta \Longrightarrow f(x) > r\big)$.

(9) $\lim\limits_{x \to a^+} f(x) = -\infty \iff \forall r \in \mathbf{R}\ \exists \delta > 0\ \forall x \in A\ \big(a < x < a + \delta \Longrightarrow f(x) < r\big)$.

When A is not bounded above and $b \in \mathbf{R}$, we make the following definitions.

(10) $\lim\limits_{x\to\infty} f(x) = b \iff \forall \epsilon > 0\ \exists m \in \mathbf{R}\ \forall x \in A\ \big(x \geq m \Longrightarrow |f(x) - b| < \epsilon\big)$.

(11) $\lim\limits_{x\to\infty} f(x) = \infty \iff \forall r \in \mathbf{R}\ \exists m \in \mathbf{R}\ \forall x \in A\ \big(x \geq m \Longrightarrow f(x) > r\big)$.

(12) $\lim\limits_{x\to\infty} f(x) = -\infty \iff \forall r \in \mathbf{R}\ \exists m \in \mathbf{R}\ \forall x \in A\ \big(x \geq m \Longrightarrow f(x) < r\big)$.

When A is not bounded below and $b \in \mathbf{R}$, we make the following definitions.

(13) $\lim\limits_{x\to-\infty} f(x) = b \iff \forall \epsilon > 0\ \exists m \in \mathbf{R}\ \forall x \in A\ \big(x \leq m \Longrightarrow |f(x) - b| < \epsilon\big)$.

(14) $\lim\limits_{x\to-\infty} f(x) = \infty \iff \forall r \in \mathbf{R}\ \exists m \in \mathbf{R}\ \forall x \in A\ \big(x \leq m \Longrightarrow f(x) > r\big)$.

(15) $\lim\limits_{x\to-\infty} f(x) = -\infty \iff \forall r \in \mathbf{R}\ \exists m \in \mathbf{R}\ \forall x \in A\ \big(x \leq m \Longrightarrow f(x) < r\big)$.

3.4 Example: Let $f(x) = \dfrac{x^2 + 2x - 3}{x^2 - 1}$. Show that $\lim\limits_{x\to 1} f(x) = 2$.

Solution: Note that for $x \neq 1$ we have

$|f(x) - 2| = \left|\frac{x^2+2x-3}{x^2-1} - 2\right| = \left|\frac{(x+3)(x-1)}{(x+1)(x-1)} - 2\right| = \left|\frac{x+3}{x+1} - 2\right| = \left|\frac{x+3-2x-2}{x+1}\right| = \left|\frac{-x+1}{x+1}\right| = \frac{|x-1|}{|x+1|}$.

Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon\}$. Let $0 < |x - 1| < \delta$. Since $0 < |x - 1|$ we have $x \neq 1$ so, as shown above, $|f(x) - 2| = \frac{|x-1|}{|x+1|}$. Since $|x - 1| < \delta \leq 1$ we have $0 < x < 3$ so that $1 < x + 1$, and hence $|f(x) - 2| = \frac{|x-1|}{|x+1|} < |x - 1|$. Finally, since $|x - a| < \delta \leq \epsilon$ we have $|f(x) - 2| \leq |x - 1| < \epsilon$. Thus $\lim\limits_{x\to 1} f(x) = 2$.

3.5 Theorem: *(Two Sided Limits) Let $A \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$ and let $a \in \mathbf{R}$. Suppose that a is a limit point of A both from the left and from the right. Then for $u \in \widehat{\mathbf{R}}$ we have $\lim\limits_{x\to a} f(x) = u$ if and only if $\lim\limits_{x\to a^-} f(x) = u = \lim\limits_{x\to a^+} f(x)$.*

Proof: We prove the theorem in the case that $u = b \in \mathbf{R}$. Suppose that $\lim\limits_{x\to a} f(x) = b \in \mathbf{R}$. Let $\epsilon > 0$. Choose $\delta > 0$ so that for all $x \in A$, if $0 < |x - a| < \delta$ then $|f(x) - b| < \epsilon$. For $x \in A$ with $a - \delta < x < a$ we have $0 < |x - a| < \delta$ and so $|f(x) - b| < \epsilon$. This shows that $\lim\limits_{x\to a^-} f(x) = b$. For $x \in A$ with $a < x < x + \delta$ we have $0 < |x - a| < \delta$ and so $|f(x) - b| < \epsilon$. This show that $\lim\limits_{x\to a^+} f(x) = b$.

Conversely, suppose that $\lim\limits_{x\to a^-} f(x) = b = \lim\limits_{x\to a^+} f(x)$. Let $\epsilon > 0$. Since $f(x) \to b$ as $x \to a^-$, we can choose $\delta_1 > 0$ so that for all $x \in A$ with $a - \delta < a < a$ we have $|f(x) - b| < \epsilon$. Since $f(x) \to b$ as $x \to a^+$ we can choose $\delta_2 > 0$ so that for all $x \in A$ with $a < x < a + \delta_2$ we have $|f(x) - b| < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Let $x \in A$ with $0 < |x - a| < \delta$. Either we have $x < a$ or we have $x > a$. In the case that $x < a$ we have $a - \delta_1 \leq a - \delta < x < a$ and so $|f(x) - b| < \epsilon$ (by the choice of δ_1). In the case that $x > a$ we have $a < x < a + \delta \leq a + \delta_2$ and so $|f(x) - b| < \epsilon$ (by the choice of δ_2). In either case we have $|f(x) - b| < \epsilon$, and so $\lim\limits_{x\to a} f(x) = b$, as required.

3.6 Remark: For the sequence $(x_k)_{k\geq p}$ in $\mathbf{R}$ given by $x_k = f(k)$ where $f : \mathbf{Z}_{\geq p} \to \mathbf{R}$, the definitions (10), (11) and (12) agree with our definitions for limits of sequences. Thus limits of sequences are a special case of limits of functions. The following theorem shows that limits of functions are determined by limits of sequences.

3.7 Theorem: *(The Sequential Characterization of Limits of Functions) Let $A \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$, and let $u \in \widehat{\mathbf{R}}$.*
(1) When $a \in \mathbf{R}$ is a limit point of A, $\lim\limits_{x\to a} f(x) = u$ if and only if for every sequence (x_k) in $A \setminus \{a\}$ with $x_k \to a$ we have $f(x_k) \to u$.
(2) When a is a limit point of A from below, $\lim\limits_{x\to a^-} f(x) = u$ if and only if for every sequence (x_k) in $\{x \in A | x < a\}$ with $x_k \to a$ we have $f(x_k) \to u$.
(3) When a is a limit point of A from above, $\lim\limits_{x\to a^+} f(x) = u$ if and only if for every sequence (x_k) in $\{x \in A | x > a\}$ with $x_k \to a$ we have $f(x_k) \to u$.
(4) When A is not bounded above, $\lim\limits_{x\to\infty} f(x) = u$ if and only if for every sequence (x_k) in A with $x_k \to \infty$ we have $f(x_k) \to u$.
(5) When A is not bounded below, $\lim\limits_{x\to-\infty} f(x) = u$ if and only if for every sequence (x_k) in A with $x_k \to -\infty$ we have $f(x_k) \to u$.

Proof: We prove Part 1 in the case that $u = b \in \mathbf{R}$. Let $a \in \mathbf{R}$ be a limit point of A. Suppose that $\lim\limits_{x\to a} f(x) = b \in \mathbf{R}$. Let (x_k) be a sequence in $A \setminus \{a\}$ with $x_k \to a$. Let $\epsilon > 0$. Since $\lim\limits_{x\to a} f(x) = b$, we can choose $\delta > 0$ so that $0 < |x - a| < \delta \Longrightarrow |f(x) = b| < \epsilon$. Since $x_k \to a$ we can choose $m \in \mathbf{Z}$ so that $k \geq m \Longrightarrow |x_k - a| < \delta$. Then for $k \geq m$, we have $|x_k - a| < \delta$ and we have $x_k \neq a$ (since the sequence (x_k) is in the set $A \setminus \{a\}$) so that $0 < |x - a| < \delta$ and hence $|f(x_k) - b| < \epsilon$. This shows that $f(x_k) \to b$.

Conversely, suppose that $\lim\limits_{x\to a} f(x) \neq b$. Choose $\epsilon_0 > 0$ so that for all $\delta > 0$ there exists $x \in A$ with $0 < |x - a| < \delta$ and $|f(x) - b| \geq \epsilon_0$. For each $k \in \mathbf{Z}^+$, choose $x_k \in A$ with $0 < |x_k - a| \leq \frac{1}{k}$ and $|f(x_k) - b| \geq \epsilon_0$. In this way we obtain a sequence $(x_k)_{k\geq 1}$ in $A \setminus \{a\}$. Since $|x_k - a| \leq \frac{1}{k}$ for all $k \in \mathbf{Z}^+$, it follows that $x_k \to a$ (indeed, given $\epsilon > 0$ we can choose $m \in \mathbf{Z}$ with $m > \frac{1}{\epsilon}$ and then $k \geq m \Longrightarrow |x_k - a| \leq \frac{1}{k} \leq \frac{1}{m} < \epsilon$). Since $|f(x_k) - b| \geq \epsilon_0$ for all k, it follows that $f(x_k) \not\to b$ (indeed if we had $f(x_k) \to b$ we could choose $m \in \mathbf{Z}$ so that $k \geq m \Longrightarrow |f(x_k) - b| < \epsilon_0$ and then we could choose $k = m$ to get $|f(x_k) - b| < \epsilon_0$).

3.8 Remark: It follows from the Sequential Characterization of Limits of Functions that all of our theorems about limits of sequences imply analogous theorems in the more general setting of limits of functions. We list several of those theorems and give one sample proof.

3.9 Theorem: *(Local Determination of Limits) Let $A \subseteq B \subseteq \mathbf{R}$, let a be a limit point of A (hence also of B) and let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ with $f(x) = g(x)$ for all $x \in A$. Then if $\lim\limits_{x\to a} g(x) = u \in \widehat{\mathbf{R}}$ then $\lim\limits_{x\to a} f(x) = u$.*
Similar results holds for limits $x \to a^\pm$ and $x \to \pm\infty$.

3.10 Theorem: *(Uniqueness of Limits) Let $A \subseteq \mathbf{R}$, let a be a limit point of A, and let $f : A \to \mathbf{R}$. For $u, v \in \widehat{\mathbf{R}}$, if $\lim\limits_{x\to a} f(x) = u$ and $\lim\limits_{x\to a} f(x) = v$ then $u = v$. Similar results hold for limits $x \to a^\pm$ and $x \to \pm\infty$.*

3.11 Theorem: *(Basic Limits) Let F be a subfield of $\mathbf{R}$, and let $A \subseteq F$. For the constant function $f : A \to F$ given by $f(x) = b$ for all $x \in A$, we have*

$$\lim_{x\to a} f(x) = b\ ,\ \lim_{x\to a^+} f(x) = b\ ,\ \lim_{x\to a^-} f(x) = b\ ,\ \lim_{x\to\infty} f(x) = b \ \text{ and } \lim_{x\to-\infty} f(x) = b,$$

and for the identity function $f : A \to F$ given by $f(x) = x$ for all $x \in A$ we have

$$\lim_{x\to a} f(x) = a\ ,\ \lim_{x\to a^+} f(x) = a\ ,\ \lim_{x\to a^-} f(x) = a\ ,\ \lim_{x\to\infty} f(x) = \infty \ \text{ and } \lim_{x\to-\infty} f(x) = -\infty$$

whenever the limits are defined.

3.12 Theorem: *(Extended Operations on Limits) Let $A \subseteq \mathbf{R}$, let $f, g : A \to \mathbf{R}$ and let a be a limit point of A. Let $u, v \in \widehat{\mathbf{R}}$ and suppose that $\lim\limits_{x\to a} f(x) = u$ and $\lim\limits_{x\to a} g(x) = v$. Then*

(1) if $u + v$ is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{x\to a}(f + g)(x) = u + v$,

(2) if $u - v$ is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{x\to a}(f - g)(x) = u - v$,

(3) if $u \cdot v$ is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{x\to a}(fg)(x) = u \cdot v$, and

(4) if u/v is defined in $\widehat{\mathbf{R}}$ then $\lim\limits_{x\to a}(f/g)(x) = u/v$.

Similar results hold for limits $x \to a^{\pm}$ and $x \to \pm\infty$.

Proof: We prove Part 4. Suppose that u/v is defined in $\widehat{\mathbf{R}}$. Let (x_k) be any sequence in $A \setminus \{a\}$ with $x_k \to a$. By the Sequential Characterization of Limits, since $\lim\limits_{x\to a} f(x) = u$ we have $f(x_k) \to u$, and since $\lim\limits_{x\to a} g(x) = v$ we have $f(x_k) \to v$. By Extended Operations on Limits of Sequences (Theorem 1.14), since $f(x_k) \to u$ and $g(x_k) \to v$ and u/v is defined in $\widehat{\mathbf{R}}$, we have $(f/g)(x_k) = \frac{f(x_k)}{g(x_k)} \to u/v$. Thus $(f/g)(x_k) \to u/v$ for every sequence (x_k) in $A \setminus \{a\}$ with $x_k \to a$. By the Sequential Characterization of Limits, it follows that $\lim\limits_{x\to a}(f/g)(x) = u/v$.

3.13 Theorem: *(The Comparison Theorem) Let $A \subseteq F$, let $f, g : A \to \mathbf{R}$ and let $a \in \mathbf{R}$ be a limit point of A. Suppose that $f(x) \le g(x)$ for all $x \in A$. Then*

(1) if $\lim\limits_{x\to a} f(x) = u$ and $\lim\limits_{x\to a} f(x) = v$ with $u, v \in \widehat{\mathbf{R}}$, then $u \le v$,

(2) if $\lim\limits_{x\to a} f(x) = \infty$ then $\lim\limits_{x\to a} g(x) = \infty$, and

(3) if $\lim\limits_{x\to a} g(x) = -\infty$ then $\lim\limits_{x\to a} g(x) = -\infty$.

Similar results hold for limits $x \to a^{\pm}$ and $x \to \pm\infty$.

3.14 Theorem: *(The Squeeze Theorem) Let $A \subseteq \mathbf{R}$, let $f, g, h : A \to \mathbf{R}$, and let $a \in \mathbf{R}$ be a limit point of A.*

(1) If $f(x) \le g(x) \le h(x)$ for all $x \in A$ and $\lim\limits_{x\to a} f(x) = b = \lim\limits_{x\to a} h(x)$, then $\lim\limits_{x\to a} g(x) = b$.

(2) If $|f(x)| \le g(x)$ for all $x \in A$ and $\lim\limits_{x\to a} g(x) = 0$ then $\lim\limits_{x\to a} f(x) = 0$.

Similar results hold for limits $x \to a^{\pm}$ and $x \to \pm\infty$.

3.15 Definition: Let $A \subseteq \mathbf{R}$, and let $f : A \to \mathbf{R}$. For $a \in A$, we say that f is **continuous** at a when

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in A \;\; \big(|x-a| < \delta \Longrightarrow |f(x) - f(a)| < \epsilon\big).$$

We say that f is **continuous** (on A) when f is continuous at every point $a \in A$.

3.16 Theorem: *Let $A \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$ and let $a \in A$. Then*
(1) if a is not a limit point of A then f is continuous at a, and
(2) if a is a limit point of A then f is continuous at a if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

Proof: The proof is left as an exercise.

3.17 Theorem: *(The Sequential Characterization of Continuity) Let $A \subseteq \mathbf{R}$, let $a \in A$, and let $f : A \to \mathbf{R}$. Then f is continuous at a if and only if for every sequence (x_k) in A with $x_k \to a$ we have $f(x_k) \to f(a)$.*

Proof: Suppose that f is continuous at a. Let (x_k) be a sequence in A with $x_k \to a$. Let $\epsilon > 0$. Choose $\delta > 0$ so that for all $x \in A$ we have $|x - a| < \delta \Longrightarrow |f(x) - f(a)| < \epsilon$. Choose $m \in \mathbf{Z}$ so that for all indices k we have $k \geq m \Longrightarrow |x_k - a| < \delta$. Then when $k \geq m$ we have $|x_k - a| < \delta$ and hence $|f(x_k) - f(a)| < \epsilon$. Thus we have $f(x_k) \to f(a)$.

Conversely, suppose that f is not continuous at a. Choose $\epsilon_0 > 0$ so that for all $\delta > 0$ there exists $x \in A$ with $|x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon_0$. For each $k \in \mathbf{Z}^+$, choose $x_k \in A$ with $|x_k - a| \leq \frac{1}{k}$ and $|f(x_k) - f(a)| \geq \epsilon_0$. Consider the sequence (x_k) in A. Since $|x_k - a| \leq \frac{1}{k}$ for all $k \in \mathbf{Z}^+$, it follows that $x_k \to a$. Since $|f(x_k) - f(a)| \geq \epsilon_0$ for all $k \in \mathbf{Z}^+$, it follows that $f(x_k) \not\to f(a)$.

3.18 Theorem: *(Operations on Continuous Functions) Let $A \subseteq \mathbf{R}$, let $f, g : A \to \mathbf{R}$, let $a \in A$ and let $c \in \mathbf{R}$. Suppose that f and g are continuous at a. Then the functions cf, $f + g$, $f - g$ and fg are all continuous at a, and if $g(a) \neq 0$ then the function f/g is continuous at a.*

Proof: The proof is left as an exercise.

3.19 Theorem: *(Composition of Continuous Functions) Let $A, B \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$, and let $h = g \circ f : C \to \mathbf{R}$ where $C = A \cap f^{-1}(B)$.*
(1) If f is continuous at $a \in C$ and g is continuous at $f(a)$, then h is continuous at a.
(2) If f is continuous (on A) and g is continuous (on B) then h is continuous (on C).

Proof: Note that Part 2 follows immediately from Part 1, so it suffices to prove Part 1. Suppose that f is continuous at $a \in A$ and g is continuous at $b = f(a) \in B$. Let (x_k) be a sequence in C with $x_k \to a$. Since f is continuous at a, we have $f(x_k) \to f(a) = b$ by the Sequential Characterization of Continuity. Since $\big(f(x_k)\big)$ is a sequence in B with $f(x_k) \to b$ and since g is continuous at b, we have $g(f(x_k)) \to g(b)$ by the Sequential Characterization of Continuity. Thus we have $h(x_k) = g\big(f(x_k)\big) \to g(b) = g\big(f(a)\big) = h(a)$. We have shown that for every sequence (x_k) in C with $x_k \to a$ we have $h(x_k) \to h(a)$. Thus h is continuous at a by the Sequential Characterization of Continuity.

3.20 Theorem: *(Functions Acting on Limits) Let $A, B \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$, let $g : B \to \mathbf{R}$ and let $h = g \circ f : C \to F$ where $C = A \cap f^{-1}(B)$. Let a be a limit point of C (hence also of A) and let b be a limit point of B. Suppose that $\lim_{x \to a} f(x) = a$ and $\lim_{y \to b} g(y) = c$. Suppose either that $f(x) \neq b$ for all $x \in C \setminus \{a\}$ or that g is continuous at $b \in B$. Then $\lim_{x \to a} h(x) = c$.*

Analogous results hold, dealing with limits $x \to a^{\pm}$, $x \to \pm\infty$, $y \to b^{\pm}$ and $y \to \pm\infty$.

Proof: The proof is left as an exercise. It is similar to the proof of the Composition of Continuous Functions Theorem.

3.21 Definition: The functions 1, x, $\sqrt[n]{x}$ with $n \in \mathbf{Z}^+$, e^x, $\ln x$, $\sin x$ and $\sin^{-1} x$, are called the **basic elementary functions**. An **elementary function** is any function $f : A \subseteq \mathbf{R} \to \mathbf{R}$ which can be obtained from the basic elementary functions using (finitely many applications of) the operations cf, $f + g$, $f - g$, $f \cdot g$, f/g and $f \circ g$.

3.22 Example: Each of the following functions $f(x)$ is elementary: $f(x) = |x| = \sqrt{x^2}$, $f(x) = \cos x = \sin\left(x + \frac{\pi}{2}\right)$, $f(x) = \tan x = \frac{\sin x}{\cos x}$, $f(x) = \tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$, $f(x) = x^a = e^{a \ln x}$ where $a \in \mathbf{R}$, $f(x) = a^x = e^{x \ln a}$ where $a > 0$, and $f(x) = \frac{e^{\sqrt{x} + \sin x}}{\tan^{-1}(\ln x)}$.

3.23 Note: We shall assume familiarity with exponential, logarithmic, trigonometric and inverse trigonometric functions. In particular, we shall assume that they are known to be continuous in their domains, (and it follows that every elementary function is continuous in its domain). We shall also assume that their asymptotic behaviour, the intervals on which they are increasing and decreasing, and all of their usual algebraic identities are known. A review of this material can be found in Chapter 1.

A rigorous proof that these basic elementary functions are continuous, and that they satisfy their usual well-known properties, is quite long and difficult (and we shall not give a proof in this course). The main difficulty lies in giving a rigorous definition for each of the basic elementary functions. In most calculus courses, we define exponential and trigonometric functions informally. We might define the function $f(x) = e^x$ to be the function with $f(0) = 1$ which is equal to its own derivative, but we do not ever prove rigorously that such a function actually exists. We might define the sine and cosine functions by saying that for $\theta > 0$, when we start at $(1, 0)$ and travel a distance θ units counterclockwise around the unit circle $x^2 + y^2 = 1$, the point at which we arrive is (by definition) the point $(x, y) = (\cos\theta, \sin\theta)$, but we have not yet rigorously defined the meaning of distance along a curve. We use these informal definitions to argue, informally, that $\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$ and then we argue that because e^x, $\sin x$ and $\cos x$ are differentiable, therefore they must be continuous.

There are various possible ways to define exponential and trigonometric functions rigorously. One way is to wait until one has rigorously defined power series and then define

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \ , \ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^n \ , \ \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n .$$

If we define e^x, $\sin x$ and $\cos x$ using these formulas, then one can prove (rigorously) that they are differential and continuous, and one can verify (although it is quite time consuming to do so) that they satisfy all of their usual well-known properties.

3.24 Example: For each of the following sequences $(x_k)_{k\geq 0}$, evaluate $\lim\limits_{k\to\infty} x_k$ if it exists.

(a) $x_k = \frac{\sqrt{3k^2+1}}{k+2}$ (b) $x_k = \frac{1+3k}{\sqrt[3]{2+k-k^2}}$ (c) $x_k = \sin^{-1}\left(k - \sqrt{k^2+k}\right)$

Solution: For Part (a), we have $x_k = \frac{\sqrt{3k^2+1}}{k+2} = \frac{\sqrt{3+\frac{1}{k^2}}}{1+2\cdot\frac{1}{k}} \longrightarrow \frac{\sqrt{3+0}}{1+2\cdot 0} = \sqrt{3}$ where we used Basic Limits, Extended Operations on Limits, the fact that $\sqrt{x}$ is continuous, and the Sequential Characterization of Limits $\big($since $\sqrt{x}$ is continuous at 3 we have $\lim\limits_{x\to 3}\sqrt{x} = \sqrt{3}$, and since $3+\frac{1}{k^2}\to 3$ we have $\lim\limits_{k\to\infty}\sqrt{3+\frac{1}{k^2}} = \lim\limits_{x\to 3}\sqrt{x} = \sqrt{3}$ by the Sequential Characterization of Limits$\big)$.

For Part (b), $x_k = \frac{1+3k}{\sqrt[3]{2+k-k^2}} = \frac{\frac{1}{k}+3}{\sqrt[3]{\frac{2}{k^2}+\frac{1}{k}-1}}\cdot k^{1/3} \longrightarrow \frac{0+3}{\sqrt[3]{0+0-1}}\cdot\infty = -1\cdot\infty = -\infty$ where we used Basic Limits, Extended Operations, the continuity of $\sqrt[3]{x}$, and the Sequential Characterization of Limits

For Part (c), note that $k-\sqrt{k^2+k} = \frac{k^2-(k^2+k)}{k+\sqrt{k^2+k}} = \frac{-k}{k+\sqrt{k^2+k}} = \frac{-1}{1+\sqrt{1+\frac{1}{k}}} \longrightarrow \frac{-1}{1+\sqrt{1+0}} = -\frac{1}{2}$, and so $x_k = \sin^{-1}\left(k-\sqrt{k^2-k}\right) \longrightarrow \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

3.25 Exercise: Evaluate each of the following limits, if they exist.

(a) $\lim\limits_{x\to 3}\dfrac{\sqrt{x+1}-2}{3-x}$ (b) $\lim\limits_{x\to 1}\sin^{-1}\left(\dfrac{2}{x-1}-\dfrac{x+3}{x^2-1}\right)$ (c) $\lim\limits_{x\to 0} e^{-1/x^2}$

(d) $\lim\limits_{x\to\infty}\dfrac{(3x+1)\sqrt{x}}{\sqrt{4x^3-2x+1}}$ (e) $\lim\limits_{x\to 1^-}\dfrac{\sqrt{x^3-2x^2+x}}{x^2+2x-3}$ (f) $\lim\limits_{x\to -1^+}\dfrac{x^2-2x-3}{x^3+4x^2+5x+2}$

3.26 Theorem: *(Intermediate Value Theorem) Let I be an interval in $\mathbf{R}$ and let $f: I\to\mathbf{R}$ be continuous. Let $a,b\in I$ with $a\leq b$ and let $y\in\mathbf{R}$. Suppose that either $f(a)\leq y\leq f(b)$ or $f(b)\leq y\leq f(a)$. Then there exists $x\in[a,b]$ with $f(x)=y$.*

Proof: Like the Monotone Convergence Theorem, the statement of this theorem is intuitively reasonable, but it is quite difficult to prove, and in most calculus courses this theorem is accepted axiomatically, without proof.

3.27 Example: Prove that there exists $x\in[0,1]$ such that $3x-x^3=1$.

Solution: Let $f(x)=3x-x^3$. Note that f is continuous (it is an elementary function) with $f(0)=0$ and $f(1)=2$ and so, by the Intermediate Value Theorem, there exists $x\in[0,1]$ such that $f(x)=1$. We remark that in fact $f(x)=1$ when $x=2\cos\left(\frac{2\pi}{9}\right)$.

3.28 Definition: Let $A\subseteq\mathbf{R}$, and let $f: A\to\mathbf{R}$. For $a\in A$, if $f(a)\geq f(x)$ for every $x\in A$, then we say that $f(a)$ is the **maximum value** of f and that f attains its maximum value at a. Similarly for $b\in A$, if $f(b)\leq f(x)$ for every $x\in A$ then we say that $f(b)$ is the **minimum value** of f (in A) and that f attains its minimum value at b. We say that f attains its **extreme values** in A when f attains its maximum value at some point $a\in A$ and f attains its minimum value at some point $b\in A$.

3.29 Theorem: *(Extreme Value Theorem) Let $a,b\in\mathbf{R}$ with $a<b$, and let $f:[a,b]\to\mathbf{R}$ be continuous. Then f attains its extreme values in $[a,b]$.*

Proof: Like the Monotone Convergence Theorem and the Intermediate Value Theorem, the statement of this theorem seems reasonable, but it is difficult to prove.

Chapter 4. Differentiation

4.1 Definition: For a subset $A \subseteq \mathbf{R}$, we say that A is **open** when it is a union of open intervals. Let $A \subseteq \mathbf{R}$ be open, let $f : A \to \mathbf{R}$. For $a \in A$, we say that f is **differentiable** at a when the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists in $\mathbf{R}$. In this case we call the limit the **derivative** of f at a, and we denote to by $f'(a)$, so we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We say that f is **differentiable** (on A) when f is differentiable at every point $a \in A$. In this case, the **derivative** of f is the function $f' : A \to \mathbf{R}$ defined by

$$f'(x) = \lim_{u \to x} \frac{f(u) - f(x)}{u - x}.$$

When f' is differentiable at a, denote the derivative of f' at a by $f''(a)$, and we call $f''(a)$ the **second derivative** of f at a. When $f''(a)$ exists for every $a \in A$, we say that f is **twice differentiable** (on A), and the function $f'' : A \to \mathbf{R}$ is called the **second derivative** of f. Similarly, $f'''(a)$ is the derivative of f'' at a and so on. More generally, for any function $f : A \to \mathbf{R}$, we define its **derivative** to be the function $f' : B \to \mathbf{R}$ where $B = \left\{a \in A \middle| f \text{ is differentiable at } a\right\}$, and we define its **second derivative** to be the function $f'' : C \to \mathbf{R}$ where $C = \left\{a \in B \middle| f' \text{ is differentiable at } a\right\}$ and so on.

4.2 Remark: Note that

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

To be precise, the limit on the left exists in $\mathbf{R}$ if and only if the limit on the right exists in $\mathbf{R}$, and in this case the two limits are equal.

4.3 Note: Let $A \subseteq \mathbf{R}$ be open, let $f : A \to \mathbf{R}$, and let $a \in A$. Then

$$\begin{aligned}
&f \text{ is differentiable at } a \text{ with derivative } f'(a) \iff \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) \\
&\iff \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \left(0 < |x - a| < \delta \implies \left|\frac{f(x) - f(a)}{x - a} - f'(a)\right| < \epsilon\right) \\
&\iff \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \left(0 < |x - a| < \delta \implies \left|\frac{f(x) - f(a) - f'(a)(x - a)}{x - a}\right| < \epsilon\right) \\
&\iff \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \left(0 < |x - a| < \delta \implies \big|f(x) - f(a) - f'(a)(x - a)\big| < \epsilon\,|x - a|\right)
\end{aligned}$$

We can also simplify this last expression a little bit by noting that when $x = a$ we have $\big|f(x) - f(a) - f'(a)(x - a)\big| = 0 = \epsilon\,|x - a|$, so we can replace inequalities by equalities and say that f is differentiable at a if and only if

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in A \left(|x - a| \le \delta \implies \big|f(x) - l(x)\big| \le \epsilon\,|x - a|\right)$$

where $l : \mathbf{R} \to \mathbf{R}$ is given by $l(x) = f(a) + f'(a)(x - a)$.

4.4 Definition: When $f : A \to \mathbf{R}$ is differentiable at a with derivative $f'(a)$, the function

$$l(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a. Note that the graph $y = l(x)$ of the linearization is the line through the point $(a, f(a))$ with slope $f'(a)$. This line is called the **tangent line** to the graph $y = f(x)$ at the point $(a, f(a))$.

4.5 Theorem: *(Differentiability Implies Continuity) Let $A \subseteq \mathbf{R}$ be open, let $f : A \to \mathbf{R}$ and let $a \in A$. If f is differentiable at a then f is continuous at a.*

Proof: We have

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a) \longrightarrow f'(a) \cdot 0 = 0 \ \text{ as } x \to a$$

and so

$$f(x) = \big(f(x) - f(a)\big) + f(a) \longrightarrow 0 + f(a) = f(a) \ \text{ as } x \to a.$$

This proves that f is continuous at a.

4.6 Theorem: *(Local Determination of the Derivative) Let $A, B \subseteq \mathbf{R}$ be open with $A \subseteq B$, let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ wih $f(x) = g(x)$ for all $x \in A$. and let $a \in A$. Then f is differentiable at a if and only if g is differentiable at a and, in this case, $f'(a) = g'(a)$.*

Proof: The proof is left as an exercise.

4.7 Theorem: *(Operations on Derivatives) Let $A \subseteq \mathbf{R}$ be open, let $f, g : A \to \mathbf{R}$, let $a \in A$, and let $c \in \mathbf{R}$. Suppose that f and g are differentiable at a. Then*

(1) (Linearity) the functions cf, $f + g$ and $f - g$ are differentiable at a with

$$(cf)'(a) = c\, f'(a) \ , \ (f + g)'(a) = f'(a) + g'(a) \ , \ (f - g)'(a) - f'(a) - g'(a),$$

(2) (Product Rule) the function fg is differentiable at a with

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a),$$

(3) (Reciprocal Rule) if $g(a) \neq 0$ then the function $1/g$ is differentiable at a with

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2},$$

(4) (Quotient Rule) if $g(a) \neq 0$ then the function f/g is differentiable at a with

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Proof: We prove Parts (2), (3) and (4). For $x \in A$ with $x \neq a$, we have

$$\begin{aligned}
\frac{(fg)(x) - (fg)(a)}{x - a} &= \frac{f(x)g(x) - f(a)g(a)}{x - a} \\
&= \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x - a} \\
&= f(x) \cdot \frac{g(x) - g(a)}{x - a} + g(a) \cdot \frac{f(x) - f(a)}{x - a} \\
&\longrightarrow f(a) \cdot g'(a) + g(a) \cdot f'(a) \ \text{ as } x \to a.
\end{aligned}$$

Note that $f(x) \to f(a)$ as $x \to a$ because f is continuous at a since differentiability implies continuity. This proves the Product Rule.

Suppose that $g(a) \neq 0$. Since g is continuous at a (because differentiability implies continuity) we can choose $\delta > 0$ so that $|x-a| \leq \delta \Longrightarrow |g(x)-g(a)| \leq \frac{|g(a)|}{2}$ and then when $|x-a| \leq \delta$ we have $|g(x)| \geq \frac{|g(a)|}{2}$ so that $g(x) \neq 0$. For $x \in A$ with $|x-a| \leq \delta$ we have

$$\frac{\left(\frac{1}{g}\right)(x) - \left(\frac{1}{g}\right)(a)}{x - -a} = \frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x-a} = \frac{-1}{g(x)g(a)} \cdot \frac{g(x)-g(a)}{x-a} \longrightarrow \frac{-1}{g(a)^2} \cdot g'(a)$$

as $x \to a$. This Proves the Reciprocal Rule.

Finally, note that Part (4) follows from Parts (2) and (3). Indeed when $g(a) \neq 0$, we have

$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \left(f \cdot \frac{1}{g}\right)'(a) = f'(a) \cdot \left(\frac{1}{g}\right)(a) + f(a) \cdot \left(\frac{1}{g}\right)'(a) \\ &= f'(a) \cdot \frac{1}{g(a)} + f(a) \cdot \frac{-g'(a)}{g(a)^2} = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.\end{aligned}$$

4.8 Theorem: *(Chain Rule) Let $A, B \subseteq \mathbf{R}$ be open, let $f : A \to \mathbf{R}$, let $g : B \to \mathbf{R}$ and let $h = g \circ f : C \to \mathbf{R}$ where $C = A \cap f^{-1}(B)$. Let $a \in C$ and let $b = f(a) \in B$. Suppose that f is differentiable at a and g is differentiable at b. Then h is differentiable at a with*

$$h'(a) = g'\big(f(a)\big)\, f'(a).$$

Proof: We provide an explanation which can be converted (with a bit of trouble) into a rigorous proof. When $x \in A$ with $x \neq a$ and $y = f(x) \in B$ wih $y \neq b$ we have

$$\begin{aligned}\frac{h(x)-h(a)}{x-a} &= \frac{g\big(f(x)\big) - g\big(f(a)\big)}{x-a} = \frac{g(y)-g(b)}{x-a} \\ &= \frac{g(y)-g(b)}{y-b} \cdot \frac{y-b}{x-a} = \frac{g(y)-g(b)}{y-b} \cdot \frac{f(x)-f(a)}{x-a} \\ &\longrightarrow g'(b) \cdot f'(a) = g'\big(f(a)\big) \cdot f'(a) \quad \text{as} \quad x \to a\end{aligned}$$

because as $x \to a$, since f is continuous at a we also have $f(x) \to f(a)$, that is $y \to b$.

We remark that when one tries to make this argument rigorous, using the ϵ-δ definition of limits, a difficulty arises because $x \neq a$ does not imply that $y \neq b$.

4.9 Definition: Recall that when $f : A \subseteq \mathbf{R} \to \mathbf{R}$, we say that f is **nondecreasing** (on A when for all $x, y \in A$, if $x \leq y$ then $f(x) \leq f(y)$, we say that f is (strictly) **increasing** (on A) when for all $x, y \in A$, if $x < y$ then $f(x) < f(y)$, we say that f is (strictly) **decreasing** (on A) when for all $x, y \in A$, if $x < y$ then $f(x) > f(y)$, and we say that f is (strictly) **monotonic** (on A) when either f is strictly increasing on A or f is strictly decreasing on A.

4.10 Theorem: *(The Inverse Function Theorem) Let I be an interval in $\mathbf{R}$, let $f : I \to \mathbf{R}$, let $J = f(I)$, and let a be a point in I which is not an endpoint.*

(1) If f is continuous then its range $J = f(I)$ is an interval in $\mathbf{R}$.
(2) If f is injective and continuous then f is strictly monotonic.
(3) If $f : I \to J$ is strictly monotonic, then so is its inverse $g : J \to I$.
(4) If f is bijective and continuous then its inverse g is continuous.

(5) If f is bijective and continuous, and f is differentiable at a with $f'(a) \neq 0$, then its inverse g is differentiable at $b = f(a)$ with $g'(b) = \frac{1}{f'(a)}$.

Proof: This theorem is quite difficult to prove and we omit the proof.

4.11 Theorem: *(Derivatives of the Basic Elementary Functions) The basic elementary functions have the following derivatives.*

(1) $(x^a)' = a\,x^{a-1}$ *where* $a \in \mathbf{R}$ *and* $x \in \mathbf{R}$ *is such that* x^{a-1} *is defined,*

(2) $(a^x)' = \ln a \cdot a^x$ *where* $a > 0$ *and* $x \in \mathbf{R}$ *and*
$(\log_a x)' = \frac{1}{\ln a} \cdot \frac{1}{x}$ *where* $0 < a \neq 1$ *and* $x > 0$*, and in particular*
$(e^x)' = e^x$ *for all* $x \in \mathbf{R}$ *and* $(\ln x)' = \frac{1}{x}$ *for all* $x > 0$*,*

(3) $(\sin x)' = \cos x$ *and* $(\cos x)' = -\sin x$ *for all* $x \in \mathbf{R}$*, and*
$(\tan x)' = \sec^2 x$ *and* $(\sec x)' = \sec x \tan x$ *for all* $x \in \mathbf{R}$ *with* $x \neq \frac{\pi}{2} + k\pi, k \in \mathbf{Z}$*,*
$(\cot x)' = -\csc^2 x$ *and* $(\csc x)' = -\cot x \csc x$ *for all* $x \in \mathbf{R}$ *with* $x \neq \pi + k\pi, k \in \mathbf{Z}$*,*

(4) $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$ *and* $(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}$ *for* $|x| < 1$*,*
$(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$ *and* $(\csc^{-1} x)' = \frac{-1}{x\sqrt{x^2-1}}$ *for* $|x| > 1$*, and*
$(\tan^{-1} x)' = \frac{1}{1+x^2}$ *and* $(\cot^{-1} x)' = \frac{-1}{1+x^2}$ *for all* $x \in \mathbf{R}$*.*

Proof: First we prove Part 1 in the case that $a \in \mathbf{Q}$. When $n \in \mathbf{Z}^+$ and $f(x) = x^n$ we have

$$\begin{aligned}\frac{f(u)-f(x)}{u-x} = \frac{u^n - x^n}{u-x} &= \frac{(u-x)(u^{n-1} + u^{n-2}x + u^{n-3}x^2 + \cdots + x^{n-1})}{u-x} \\ &= u^{n-1} + u^{n-2}x + u^{n-3}x^2 + \cdots + x^{n-1} \longrightarrow n\,x^{n-1} \quad \text{as } u \to x.\end{aligned}$$

This shows that $(x^n)' = n\,x^{n-1}$ for all $x \in \mathbf{R}$ when $n \in \mathbf{Z}^+$. By the Reciprocal Rule, for $x \neq 0$ we have

$$(x^{-n})' = \left(\frac{1}{x^n}\right)' = -\frac{(x^n)'}{(x^n)^2} = -\frac{n\,x^{n-1}}{x^{2n}} = -n\,x^{-n-1}.$$

The function $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$ (when n is odd, $x^{1/n}$ is defined for all $x \in \mathbf{R}$, and when n is even, $x^{1/n}$ is defined only for $x \geq 0$). Since $f'(x) = (x^n)' = n\,x^{n-1}$ we have $f'(x) = 0$ when $x = 0$. By the Inverse Function Theorem, when $x \neq 0$ we have

$$(x^{1/n})' = g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n\,g(x)^{n-1}} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n\,x^{1-\frac{1}{n}}} = \tfrac{1}{n}\,x^{\frac{1}{n}-1}.$$

Finally, when $n \in \mathbf{Z}^+$ and $k \in \mathbf{Z}$ with $\gcd(k,n) = 1$, by the Chain Rule we have

$$(x^{k/n})' = \big((x^{1/n})^k\big)' = k(x^{1/n})^{k-1}(x^{1/n})' = k\,x^{\frac{k-1}{n}} \cdot \tfrac{1}{n}\,x^{\frac{1-n}{n}} = \tfrac{k}{n}\,x^{\frac{k}{n}-1}.$$

We have proven Part 1 in the case that $a \in \mathbf{Q}$.

Next we shall prove Part 2. For $f(x) = a^x$ where $a > 0$, we have

$$\frac{f(x+h)-f(x)}{h} = \frac{a^{x+h} - a^x}{h} = \frac{a^x a^h - a^x}{h} = a^x \cdot \frac{a^h - 1}{h}$$

and so we have $f'(x) = a^x\Big(\lim\limits_{h\to 0} \frac{a^h-1}{h}\Big)$ provided that the limit exists and is finite. For $g(x) = \log_a x$, where $0 < a \neq 1$ and $x > 0$, we have

$$\frac{g(x+h)-g(x)}{h} = \frac{\log_a(x+h) - \log_a x}{h} = \frac{\log_a\left(\frac{x+h}{x}\right)}{h} = \frac{\log_a\left(1+\frac{h}{x}\right)}{x \cdot \frac{h}{x}} = \tfrac{1}{x} \cdot \log_a\left(1+\tfrac{h}{x}\right)^{x/h}$$

and so we have $g'(x) = \frac{1}{x} \cdot \log_a\Big(\lim\limits_{h\to 0}\left(1+\frac{h}{x}\right)^{x/h}\Big)$ provided the limit exists and is finite.

By letting $u = \frac{h}{x}$ we see that

$$\lim_{h \to 0^+} \left(1 + \tfrac{h}{x}\right)^{x/h} = \lim_{u \to \infty} \left(1 + \tfrac{1}{u}\right)^u = e$$

by the definition of the number e. By letting $u = -\frac{h}{x}$, a similar argument shows that

$$\lim_{h \to 0^-} \left(1 + \tfrac{h}{x}\right)^{x/h} = \lim_{u \to \infty} \left(1 - \tfrac{1}{u}\right)^{-u} = e.$$

Thus the derivative $g'(x)$ does exist and we have

$$(\log_a x)' = g'(x) = \tfrac{1}{x}\, \log_a \left(\lim_{h \to 0} \left(1 + \tfrac{h}{x}\right)^{x/h} \right) = \tfrac{1}{x}\, \log_a e = \tfrac{1}{x} \cdot \tfrac{\ln e}{\ln a} = \tfrac{1}{x\, \ln a}.$$

Since $g(x) = \log_a x$ is differentiable with $g'(x) \neq 0$ it follows from the Inverse Function Theorem that $f(x) = a^x$ is differentiable with derivative

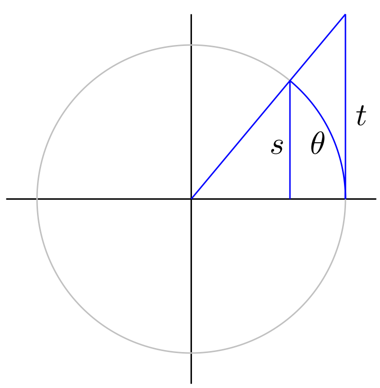
$$(a^x)' = f'(x) = \frac{1}{g'(f(x))} = \frac{1}{\frac{1}{f(x)\ln a}} = \ln a \cdot f(x) = \ln a \cdot a^x.$$

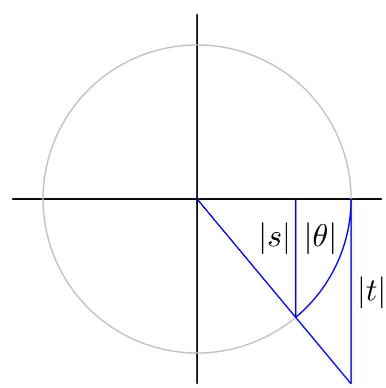
This proves Part 2.

Now we return to complete the proof of Part 1, in the case that $a \notin \mathbf{Q}$. When $a > 0$ we have $a^x = e^{x \ln a}$ for all $x > 0$ and so by the Chain Rule

$$(x^a)' = \left(e^{a \ln x}\right)' = e^{a \ln x}(a \ln x)' = x^a \cdot \tfrac{a}{x} = a\, x^{a-1}.$$

Let us move on to the proof of Part 3. We shall need two trigonometric limits which we shall explain informally (non-rigorously) with the help of pictures. Consider the following two pictures, the first showing an angle θ with $0 < \theta < \frac{\pi}{2}$ and the second with $-\frac{\pi}{2} < \theta < 0$. In both diagrams, the circle has radius 1 and $s = \sin\theta$ and $t = \tan\theta$.





In the first diagram, where $0 < \theta < \frac{\pi}{2}$, we have $\sin\theta < \theta < \tan\theta$, and dividing by $\sin\theta$ (which is positive) gives $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$. In the second diagram, where $-\frac{\pi}{2} < \theta < 0$, we have $-\sin\theta < -\theta < -\tan\theta$, and dividing by $-\sin\theta$ (which is positive) gives $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$. In either case, taking the reciprocal gives $\cos\theta < \frac{\sin\theta}{\theta} < 1$. Since $\lim\limits_{\theta \to 0} \cos\theta = \cos(0) = 1$, it follows from the Squeeze Theorem that

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

From this limit we obtain the second trigonometric limit,

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta\,(1 + \cos\theta)} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1 + \cos\theta} = 1 \cdot \tfrac{0}{2} = 0.$$

Using the above two trigonometric limits, we have

$$\begin{aligned}
(\sin x)' &= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{\sin x\cos h - \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \Big(\cos x \cdot \frac{\sin h}{h} - \sin x \cdot \frac{1-\cos h}{h} \Big) \\
&= \cos x \cdot 1 - \sin x \cdot 0 = \cos x \\
(\cos x)' &= \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0} \frac{\cos x\cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \Big(-\sin x \cdot \frac{\sin h}{h} - \cos x \cdot \frac{1-\cos h}{h} \Big) \\
&= -\sin x \cdot 1 - \cos x \cdot 0 = -\sin x.
\end{aligned}$$

By the Quotient Rule, we have

$$(\tan x)' = \Big(\frac{\sin x}{\cos x}\Big)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

We leave it as an exercise to complete the proof of Part 3 by calculating the derivatives of $\sec x$ and $\csc x$.

Finally, we shall derive the formula for $(\sin^{-1} x)'$ and leave the rest of the proof of Part 4 as an exercise. Note that by the Inverse Function Theorem (which we did not prove), the function $\sin^{-1} x$ is differentiable in $(-1,1)$. Since $\sin(\sin^{-1} x) = x$ for all $x \in (-1,1)$, we can take the derivative on both sides (using the Chain Rule on the left) to get $\cos(\sin^{-1} x) \cdot (\sin^{-1} x)' = 1$ and hence

$$(\sin^{-1} x)' = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-\sin^2(\sin^{-1} x)}} = \frac{1}{\sqrt{1-x^2}}.$$

4.12 Definition: Let $A \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$ and let $a \in A$. We say that f has a **local maximum** value at a when

$$\exists \delta > 0 \; \forall x \in A \; \Big(|x-a| \le \delta \Longrightarrow f(x) \le f(a) \Big).$$

Similarly, we say that f has a **local minimum** value at a when

$$\exists \delta > 0 \; \forall x \in A \; \Big(|x-a| \le \delta \Longrightarrow f(x) \ge f(a) \Big).$$

4.13 Theorem: *(Fermat's Theorem) Let $A \subseteq \mathbf{R}$ be open, let $f : A \to \mathbf{R}$, and let $a \in A$. Suppose that f is differentiable at a and that f has a local maximum or minimum value at a. Then $f'(a) = 0$.*

Proof: We suppose that f has a local maximum value at a (the case that f has a local minimum value at a is similar). Choose $\delta > 0$ so that $|x-a| \le \delta \Longrightarrow f(x) \le f(a)$. For $x \in A$ with $a < x < a+\delta$, since $x > a$ and $f(x) \ge f(a)$ we have $\frac{f(x)-f(a)}{x-a} \ge 0$, and so

$$f'(a) = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \ge 0$$

by the Comparison Theorem. Similarly, for $x \in A$ with $a - \delta \le x < a$, since $x < a$ and $f(x) \ge f(a)$ we have $\frac{f(x)-f(a)}{x-a} \le 0$, and so

$$f'(a) = \lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} \le 0.$$

4.14 Theorem: *(Rolle's Theorem and the Mean Value Theorem) Let $a, b \in \mathbf{R}$ with $a < b$.*

(1) (Rolle's Theorem) If $f : [a, b] \to \mathbf{R}$ differentiable in (a, b) and continuous at a and b with $f(a) = 0 = f(b)$ then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

(2) (The Mean Value Theorem) If $f : [a, b] \to \mathbf{R}$ is differentiable in (a, b) and continuous at a and b then there exists a point $c \in (a, b)$ with $f'(c)\,(b - a) = f(b) - f(a)$.

Proof: To Prove Rolle's Theorem, let $f : [a, b] \to \mathbf{R}$ be differentiable in (a, b) and continuous at a and b with $f(a) = 0 = f(b)$. If f is constant, then $f'(x) = 0$ for all $x \in [a, b]$. Suppose that f is not constant. Either $f(x) > 0$ for some $x \in (a, b)$ or $f(x) < 0$ for some $x \in (a, b)$. Suppose that $f(x) > 0$ for some $x \in (a, b)$ (the case that $f(x) < 0$ for some $x \in (a, b)$ is similar). By the Extreme Value Theorem, f attains its maximum value at some point, say $c \in [a, b]$. Since $f(x) > 0$ for some $x \in (a, b)$, we must have $f(c) > 0$. Since $f(a) = f(b) = 0$ and $f(c) > 0$, we have $c \in (a, b)$. By Fermat's Theorem, we have $f'(c) = 0$. This completes the proof of Rolle's Theorem.

To prove the Mean Value Theorem, suppose that $f : [a, b] \to \mathbf{R}$ is differentiable in (a, b) and continuous at a and b. Let $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}\,(x - a)$. Then g is differentiable in (a, b) with $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$ and g is continuous at a and b with $g(a) = 0 = g(b)$. By Rolle's Theorem, we can choose $c \in (a, b)$ so that $f'(c) = 0$, and then $g'(c) = \frac{f(b)-f(a)}{b-a}$, as required.

4.15 Corollary: *Let $a, b \in \mathbf{R}$ with $a < b$. Let $f : [a, b] \to \mathbf{R}$. Suppose that f is differentiable in (a, b) and continuous at a and b.*

(1) If $f'(x) \geq 0$ for all $x \in (a, b)$ then f is nondecreasing on $[a, b]$.
(2) If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing on $[a, b]$.
(3) If $f'(x) \leq 0$ for all $x \in (a, b)$ then f is nonincreasing on $[a, b]$.
(4) If $f'(x) < 0$ for all $x \in (a, b)$ then f is strictly decreasing on $[a, b]$.
(5) If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$.
(6) If $g : [a, b] \to \mathbf{R}$ is continuous at a and b and differentiable in (a, b) with $g'(x) = f'(x)$ for all $x \in (a, b)$, then for some $c \in \mathbf{R}$ we have $g(x) = f(x) + c$ for all $x \in (a, b)$.

Proof: We prove Part 1 and leave the rest of the proofs as exercises. Suppose that $f'(x) \geq 0$ for all $x \in (a, b)$. Let $a \leq x < y \leq b$. Choose $c \in (x, y)$ so that $f'(c) = \frac{f(y)-f(x)}{y-x}$. Then $f(y) - f(x) = f'(c)(y - x) \geq 0$ and so $f(y) \geq f(x)$. Thus f is nondecreasing on $[a, b]$.

4.16 Corollary: *(The Second Derivative Test) Let I be an interval in $\mathbf{R}$, let $f : I \to \mathbf{R}$ and let $a \in I$. Suppose that f is differentiable in I with $f'(a) = 0$.*

(1) If $f''(a) > 0$ then f has a local minimum at a.
(2) If $f''(a) < 0$ then f has a local maximum at a.

Proof: The proof is left as an exercise.

4.17 Theorem: *(l'Hôpital's Rule) Let I be a non degenerate interval in $\mathbf{R}$. Let $a \in I$, or let a be an endpoint of I. Let $f, g : I \setminus \{a\} \to \mathbf{R}$. Suppose that f and g are differentiable in $I \setminus \{a\}$ with $g'(x) \neq 0$ for all $x \in I \setminus \{a\}$. Suppose either that $\lim\limits_{x \to a} f(x) = 0 = \lim\limits_{x \to a} g(x)$ or that $\lim\limits_{x \to a} g(x) = \pm\infty$. Suppose that $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = u \in \hat{\mathbf{R}}$. Then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = u$.*

Similar results hold for limits $x \to a^+$, $x \to a^-$, $x \to \infty$ and $x \to -\infty$.

Proof: We omit the proof, which is fairly difficult.

Chapter 5. Integration

The Riemann Integral

5.1 Definition: A **partition** of the closed interval $[a,b]$ is a set $X = \{x_0, x_1, \cdots, x_n\}$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b\,.$$

The intervals $[x_{i-1}, x_i]$ are called the **subintervals** of $[a,b]$, and we write

$$\Delta_i x = x_i - x_{i-1}$$

for the size of the i^{th} subinterval. Note that

$$\sum_{i=1}^{n} \Delta_i x = b - a\,.$$

The **size** of the partition X, denoted by $|X|$ is

$$|X| = \max\left\{\Delta_i x \middle| 1 \le i \le n\right\}.$$

5.2 Definition: Let X be a partition of $[a,b]$, and let $f : [a,b] \to \mathbf{R}$ be bounded. A **Riemann sum** for f on X is a sum of the form

$$S = \sum_{i=1}^{n} f(t_i)\Delta_i x \qquad \text{for some } t_i \in [x_{i-1}, x_i]\,.$$

The points t_i are called **sample points**.

5.3 Definition: Let $f : [a,b] \to \mathbf{R}$ be bounded. We say that f is **(Riemann) integrable** on $[a,b]$ when there exists a number I with the property that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every partition X of $[a,b]$ with $|X| < \delta$ we have $|S - I| < \epsilon$ for every Riemann sum for f on X, that is

$$\left|\sum_{i=1}^{n} f(t_i)\Delta_i x - I\right| < \epsilon\,.$$

for every choice of $t_i \in [x_{i-1}, x_i]$ The number I can be shown to be unique. It is called the **(Riemann) integral** of f on $[a,b]$, and we write

$$I = \int_a^b f\ \text{, or } I = \int_a^b f(x)\,dx\,.$$

5.4 Example: Show that the constant function $f(x) = c$ is integrable on any interval $[a,b]$ and we have $\displaystyle\int_a^b c\ dx = c(b-a)$.

Solution: The solution is left as an exercise.

5.5 Example: Show that the identity function $f(x) = x$ is integrable on any interval $[a,b]$, and we have $\displaystyle\int_a^b x\ dx = \tfrac{1}{2}(b^2 - a^2)$.

Solution: Let $\epsilon > 0$. Choose $\delta = \frac{2\epsilon}{b-a}$. Let X be any partition of $[a,b]$ with $|X| < \delta$. Let $t_i \in [x_{i-1}, x_i]$ and set $S = \sum\limits_{i=1}^{n} f(t_i)\Delta_i x = \sum\limits_{i=1}^{n} t_i \Delta_i x$. We must show that $|S - \frac{1}{2}(b^2 - a^2)| < \epsilon$. Notice that

$$\begin{aligned}
\sum_{i=1}^{n}(x_i + x_{i-1})\Delta_i x = \sum_{i=1}^{n}(x_i + x_{i-1})(x_i - x_{i-1}) &= \sum_{i=1}^{n} {x_i}^2 - {x_{i-1}}^2 \\
&= ({x_1}^2 - {x_0}^2) + ({x_2}^2 - {x_1}^2) + \cdots + ({x_{n-1}}^2 - {x_{n-2}}^2) + ({x_n}^2 - {x_{n-1}}^2) \\
&= -{x_0}^2 + ({x_1}^2 - {x_1}^2) + \cdots + ({x_{n-1}}^2 - {x_{n-1}}^2) + {x_n}^2 \\
&= {x_n}^2 - {x_0}^2 = b^2 - a^2
\end{aligned}$$

and that when $t_i \in [x_{i-1}, x_i]$ we have $\left|t_i - \frac{1}{2}(x_i + x_{i-1})\right| \leq \frac{1}{2}(x_i - x_{i-1}) = \frac{1}{2}\Delta_i x$, and so

$$\begin{aligned}
\left|S - \tfrac{1}{2}(b^2 - a^2)\right| &= \left|\sum_{i=1}^{n} t_i \Delta_i x - \tfrac{1}{2}\sum_{i=1}^{n}(x_i + x_{i-1})\Delta_i x\right| \\
&= \left|\sum_{i=1}^{n}\left(t_i - \tfrac{1}{2}(x_i + x_{i+1})\right)\Delta_i x\right| \\
&\leq \sum_{i=1}^{n}\left|t_i - \tfrac{1}{2}(x_i + x_{i+1})\right|\Delta_i x \\
&\leq \sum_{i=1}^{n} \tfrac{1}{2}\Delta_i x \Delta_i x \leq \sum_{i=1}^{n} \tfrac{1}{2}\delta \Delta_i x \\
&= \tfrac{1}{2}\delta(b-a) = \epsilon\,.
\end{aligned}$$

5.6 Example: Let $f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbf{Q} \\ 0 & \text{if} \quad x \notin \mathbf{Q}\,. \end{cases}$ Show that f is not integrable on $[0,1]$.

Solution: Suppose, for a contradiction, that f is integrable on $[0,1]$, and write $I = \int_0^1 f$. Let $\epsilon = \frac{1}{2}$. Choose δ so that for every partition X with $|X| < \delta$ we have $|S - I| < \frac{1}{2}$ for every Riemann sum S for f on X. Choose a partition X with $|X| < \delta$. Let $S_1 = \sum\limits_{i=1}^{n} f(t_i)\Delta_i x$ where each $t_i \in [x_{i-1}, x_i]$ is chosen with $t_i \in \mathbf{Q}$, and let $S_2 = \sum\limits_{i=1}^{n} f(s_i)\Delta_i x$ where each $s_i \in [x_{i-1}, x_i]$ is chosen with $s_i \notin \mathbf{Q}$. Note that we have $|S_1 - I| < \frac{1}{2}$ and $|S_2 - I| < \frac{1}{2}$. Since each $t_i \in \mathbf{Q}$ we have $f(t_i) = 1$ and so $S_1 = \sum\limits_{i=1}^{n} f(t_i)\Delta_i x = \sum\limits_{i=1}^{n} \Delta_i x = 1 - 0 = 1$, and since each $s_i \notin \mathbf{Q}$ we have $f(s_i) = 0$ and so $S_2 = \sum\limits_{i=1}^{n} f(s_i)\Delta_i x = 0$. Since $|S_1 - I| < \frac{1}{2}$ we have $|1 - I| < \frac{1}{2}$ and so $\frac{1}{2} < I < \frac{3}{2}$, and since $|S_2 - I| < \frac{1}{2}$ we have $|0 - I| < \frac{1}{2}$ and so $-\frac{1}{2} < I < \frac{1}{2}$, giving a contradiction.

Integrals of Continuous Functions

5.7 Theorem: *(Continuous Functions are Integrable) Let $f : [a, b] \to \mathbf{R}$ be continuous. Then f is integrable on $[a, b]$.*

Proof: We omit the proof, which is quite difficult.

5.8 Note: Let f be integrable on $[a, b]$. Let X_n be any sequence of partitions of $[a, b]$ with $\lim\limits_{n\to\infty} |X_n| = 0$. Let S_n be any Riemann sum for f on X_n. Then $\{S_n\}$ converges with

$$\lim_{n\to\infty} S_n = \int_a^b f(x)\,dx\,.$$

Proof: Write $I = \int_a^b f$. Given $\epsilon > 0$, choose $\delta > 0$ so that for every partition X of $[a, b]$ with $|X| < \delta$ we have $|S - I| < \epsilon$ for every Riemann sum S for f on X, and then choose N so that $n > N \Longrightarrow |X_n| < \delta$. Then we have $n > N \Longrightarrow |S_n - I| < \epsilon$.

5.9 Note: Let f be integrable on $[a, b]$. If we let X_n be the partition of $[a, b]$ into n equal-sized subintervals, and we let S_n be the Riemann sum on X_n using right-endpoints, then by the above note we obtain the formula

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_{n,i})\Delta_{n,i}x \text{ , where } x_{n,i} = a + \tfrac{b-a}{n}\, i \text{ and } \Delta_{n,i}x = \tfrac{b-a}{n}\,.$$

5.10 Example: Find $\displaystyle\int_0^2 2^x\,dx$.

Solution: Let $f(x) = 2^x$. Note that f is continuous and hence integrable, so we have

$$\begin{aligned}
\int_0^2 2^x\,dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_{n,i})\Delta_{n,i}x = \lim_{n\to\infty} \sum_{i=1}^n f\left(\tfrac{2i}{n}\right)\left(\tfrac{2}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^n 2^{2i/n}\left(\tfrac{2}{n}\right) \\
&= \lim_{n\to\infty} \frac{2\cdot 4^{1/n}}{n} \cdot \frac{4-1}{4^{1/n}-1} \text{ , by the formula for the sum of a geometric sequence} \\
&= \left(\lim_{n\to\infty} 6\cdot 4^{1/n}\right)\left(\lim_{n\to\infty} \frac{1}{n\left(4^{1/n}-1\right)}\right) = 6 \lim_{n\to\infty} \frac{\frac{1}{n}}{4^{1/n}-1} = 6 \lim_{x\to 0} \frac{x}{4^x - 1} \\
&= 6 \lim_{x\to 0} \frac{1}{\ln 4 \cdot 4^x} \text{ , by l'Hôpital's Rule} \\
&= \tfrac{6}{\ln 4} = \tfrac{3}{\ln 2}\,.
\end{aligned}$$

5.11 Lemma: *(Summation Formulas) We have*

$$\sum_{i=1}^{n} 1 = n \ , \ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \ , \ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \ , \ \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Proof: These formulas could be proven by induction, but we give a more constructive proof. It is obvious that $\sum\limits_{i=1}^{n} 1 = 1+1+\cdots 1 = n$. To find $\sum\limits_{i=1}^{n} i$, consider $\sum\limits_{n=1}^{n} \big(i^2 - (i-1)^2\big)$. On the one hand, we have

$$\begin{aligned}\sum_{i=1}^{n} \big(i^2 - (i-1)^2\big) &= (1^2 - 0^2) + (2^2 - 1^2) + \cdots + ((n-1)^2 - (n-2)^2) + (n^2 - (n-1)^2) \\ &= -0^2 + (1^2 - 1^2) + (2^2 - 2^2) + \cdots + ((n-1)^2 - (n-1)^2) + n^2 \\ &= n^2\end{aligned}$$

and on the other hand,

$$\sum_{i=1}^{n} \big(i^2 - (i-1)^2\big) = \sum_{i=1}^{n} \big(i^2 - (i^2 - 2i + 1)\big) = \sum_{i=1}^{n} (2i-1) = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

Equating these gives $n^2 = 2\sum\limits_{i=1}^{n} i - \sum\limits_{i=1}^{n} 1$ and so

$$2\sum_{i=1}^{n} i = n^2 + \sum_{i=1}^{n} 1 = n^2 + n = n(n+1)\,,$$

as required. Next, to find $\sum\limits_{n=1}^{\infty} i^2$, consider $\sum\limits_{i=1} \big(i^3 - (i-1)^3\big)$. On the one hand we have

$$\begin{aligned}\sum_{i=1}^{n} \big(i^3 - (i-1)^3\big) &= (1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + \cdots + (n^3 - (n-1)^3) \\ &= -0^3 + (1^3 - 1^3) + (2^3 - 2^3) + \cdots + ((n-1)^3 - (n-1)^3) + n^3 \\ &= n^3\end{aligned}$$

and on the other hand,

$$\begin{aligned}\sum_{i=1}^{n} \big(i^3 - (i-1)^3\big) &= \sum_{i=1}^{n} \big(i^3 - (i^3 - 3i^2 + 3i - 1)\big) \\ &= \sum_{i=1}^{n} (3i^2 - 3i + 1) = 3\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\,.\end{aligned}$$

Equating these gives $n^3 = 3\sum\limits_{i=1}^{n} i^2 - 3\sum\limits_{i=1}^{n} i + \sum\limits_{i=1}^{n} 1$ and so

$$6\sum_{i=1}^{n} i^2 = 2n^3 + 6\sum_{i=1}^{n} i - 2\sum_{i=1}^{n} 1 = 2n^3 + 3n(n+1) - 2n = n(n+1)(2n+1)$$

as required. Finally, to find $\sum\limits_{i=1}^{n} i^3$, consider $\sum\limits_{i=1}^{n} \big(i^4 - (i-1)^4\big)$. On the one hand we have

$$\sum_{i=1}^{n} \big(i^4 - (i-1)^4\big) = n^4\,,$$

(as above) and on the other hand we have

$$\sum_{i=1}^{n} \big(i^4 - (i-1)^4\big) = \sum_{i=1}^{n} (4i^3 - 6i^2 + 4i - 1) = 4\sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\,.$$

Equating these gives $n^4 = 4\sum\limits_{i=1}^n i^3 - 6\sum\limits_{i=1}^n i^2 + 4\sum\limits_{i=1}^n i - \sum\limits_{i=1}^n 1$ and so

$$\begin{aligned}
4\sum_{i=1}^n i^3 &= n^4 + 6\sum_{i=1}^n i^2 - 4\sum_{i=1}^n i + \sum_{i=1}^n 1 \\
&= n^4 + n(n+1)(2n+1) - 2n(n+1) + n \\
&= n^4 + 2n^3 + n^2 = n^2(n+1)^2\,,
\end{aligned}$$

as required.

5.12 Example: Find $\displaystyle\int_1^3 x + 2x^3\ dx$.

Solution: Let $f(x) = x + 2x^3$. Then

$$\begin{aligned}
\int_1^3 x + 2x^3\ dx &= \lim_{n\to\infty}\sum_{i=1}^n f(x_{n,i})\Delta_{n,i}x \\
&= \lim_{n\to\infty}\sum_{i=1}^n f\left(1 + \tfrac{2}{n}\,i\right)\left(\tfrac{2}{n}\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^n \left(\left(1 + \tfrac{2}{n}\,i\right) + 2\left(1 + \tfrac{2}{n}\,i\right)^3\right)\left(\tfrac{2}{n}\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^n \left(1 + \tfrac{2}{n}\,i + 2\left(1 + \tfrac{6}{n}\,i + \tfrac{12}{n^2}\,i^2 + \tfrac{8}{n^3}\,i^3\right)\right)\left(\tfrac{2}{n}\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^n \left(\tfrac{6}{n} + \tfrac{28}{n^2}\,i + \tfrac{48}{n^3}\,i^2 + \tfrac{32}{n^4}\,i^3\right) \\
&= \lim_{n\to\infty}\left(\tfrac{6}{n}\sum_{i=1}^n 1 + \tfrac{28}{n^2}\sum_{i=1}^n i + \tfrac{48}{n^3}\sum_{i=1}^n i^2 + \tfrac{32}{n^4}\sum_{i=1}^n i^3\right) \\
&= \lim_{n\to\infty}\left(\tfrac{6}{n}\cdot n + \tfrac{28}{n^2}\cdot\tfrac{n(n+1)}{2} + \tfrac{48}{n^3}\cdot\tfrac{n(n+1)(2n+1)}{6} + \tfrac{32}{n^4}\cdot\tfrac{n^2(n+1)^2}{4}\right) \\
&= 6 + \tfrac{28}{2} + \tfrac{48\cdot 2}{6} + \tfrac{32}{4} = 44\,.
\end{aligned}$$

Basic Properties of Integrals

5.13 Theorem: *(Linearity) Let f and g be integrable on $[a,b]$ and let $c \in \mathbf{R}$. Then $f+g$ and cf are both integrable on $[a,b]$ and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

and

$$\int_a^b cf = c\int_a^b f\,.$$

Proof: The proof is left as an exercise.

5.14 Theorem: *(Comparison) Let f and g be integrable on $[a,b]$. If $f(x) \leq g(x)$ for all $x \in [a,b]$ then*

$$\int_a^b f \leq \int_a^b g\,.$$

Proof: The proof is left as an exercise.

5.15 Theorem: *(Additivity) Let $a < b < c$ and let $f : [a,c] \to \mathbf{R}$ be bounded. Then f is integrable on $[a,c]$ if and only if f is integrable both on $[a,b]$ and on $[b,c]$, and in this case*

$$\int_a^b f + \int_b^c f = \int_a^c f\,.$$

Proof: We omit the proof, which is quite difficult.

5.16 Definition: We define $\displaystyle\int_a^a f = 0$ and for $a < b$ we define $\displaystyle\int_b^a f = -\int_a^b f$.

5.17 Note: Using the above definition, the Additivity Theorem extends to the case that $a,b,c \in \mathbf{R}$ are not in increasing order: for any $a,b,c \in \mathbf{R}$, if f is integrable on $\big[\min\{a,b,c\}, \max\{a,b,c\}\big]$ then

$$\int_a^b f + \int_b^c f = \int_a^c f\,.$$

5.18 Theorem: *(Estimation) Let f be integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$ and*

$$\left|\int_a^b f\right| \leq \int_a^b |f|\,.$$

Proof: We omit the proof, which is quite difficult.

The Fundamental Theorem of Calculus

5.19 Notation: For a function F, defined on an interval containing $[a,b]$, we write

$$\Big[F(x)\Big]_a^b = F(b) - F(a)\,.$$

5.20 Theorem: *(The Fundamental Theorem of Calculus)*
(1) Let f be integrable on $[a,b]$. Define $F:[a,b]\to\mathbf{R}$ by

$$F(x) = \int_a^x f = \int_a^x f(t)\,dt\,.$$

Then F is continuous on $[a,b]$. Moreover, if f is continuous at a point $x\in[a,b]$ then F is differentiable at x and

$$F'(x) = f(x)\,.$$

(2) Let f be integrable on $[a,b]$. Let F be differentiable on $[a,b]$ with $F'=f$. Then

$$\int_a^b f = \Big[F(x)\Big]_a^b = F(b) - F(a)\,.$$

Proof: (1) Let M be an upper bound for $|f|$ on $[a,b]$. For $a\le x,y\le b$ we have

$$\big|F(y)-F(x)\big| = \left|\int_a^y f - \int_a^x f\right| = \left|\int_x^y f\right| \le \left|\int_x^y |f|\right| \le \left|\int_x^y M\right| = M|y-x|$$

so given $\epsilon>0$ we can choose $\delta = \frac{\epsilon}{M}$ to get

$$|y-x|<\delta \Longrightarrow \big|F(y)-F(x)\big| \le M|y-x| < M\delta = \epsilon\,.$$

Thus F is continuous on $[a,b]$. Now suppose that f is continuous at the point $x\in[a,b]$. Note that for $a\le x,y\le b$ with $x\ne y$ we have

$$\begin{aligned}
\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| &= \left|\frac{\int_a^y f - \int_a^x f}{y-x} - f(x)\right| \\
&= \left|\frac{\int_x^y f}{y-x} - \frac{\int_x^y f(x)}{y-x}\right| \\
&= \frac{1}{|y-x|}\left|\int_x^y \big(f(t)-f(x)\big)\,dt\right| \\
&\le \frac{1}{|y-x|}\left|\int_x^y \big|f(t)-f(x)\big|\,dt\right|\,.
\end{aligned}$$

Given $\epsilon>0$, since f is continuous at x we can choose $\delta>0$ so that

$$|y-x|<\delta \Longrightarrow \big|f(y)-f(x)\big| < \epsilon$$

and then for $0<|y-x|<\delta$ we have

$$\begin{aligned}
\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| &\le \frac{1}{|y-x|}\left|\int_x^y \big|f(t)-f(x)\big|\,dt\right| \\
&\le \frac{1}{|y-x|}\left|\int_x^y \epsilon\,dt\right| = \frac{1}{|y-x|}\,\epsilon|y-x| = \epsilon\,.
\end{aligned}$$

and thus we have $F'(x)=f(x)$ as required.

(2) Let f be integrable on $[a,b]$. Suppose that F is differentiable on $[a,b]$ with $F'=f$. Let $\epsilon>0$ be arbitrary. Choose $\delta>0$ so that for every partition X of $[a,b]$ with $|X|<\delta$ we have $\left|\int_a^b f-\sum_{i=1}^n f(t_i)\Delta_i x\right|<\epsilon$ for every choice of sample points $t_i\in[x_{i-1},x_i]$. Choose sample points $t_i\in[x_{i-1},x_i]$ as in the Mean Value Theorem so that

$$F'(t_i)=\frac{F(x_i)-F(x_{i-1})}{x_i-x_{i-1}}\,,$$

that is $f(t_i)\Delta_i x=F(x_i)-F(x_{i-1})$. Then $\left|\int_a^b f-\sum_{i=1}^n f(t_i)\Delta_i x\right|<\epsilon$, and

$$\begin{aligned}\sum_{i=1}^n f(t_i)\Delta_i x&=\sum_{i=1}^n\big(F(x_i)-F(x_{i-1})\big)\\&=\big(F(x_1)-F(x)\big)+\big(F(x_2)-F(x_1)\big)+\cdots+\big(F(n-1)-F(x_n)\big)\\&=-F(x)+\big(F(x_1)-F(x_1)\big)+\cdots+\big(F(x_{n-1})-F(x_{n-1})\big)+F(x_n)\\&=F(x_n)-F(x)=F(b)-F(a)\,.\end{aligned}$$

and so $\left|\int_a^b f-\big(F(b)-F(a)\big)\right|<\epsilon$. Since ϵ was arbitrary, $\left|\int_a^b f-\big(F(b)-F(a)\big)\right|=0$.

5.21 Definition: A function F such that $F'=f$ on an interval is called an **antiderivative** of f on the interval.

5.22 Note: If $G'=F'=f$ on an interval, then $(G-F)'=0$, and so $G-F$ is constant on the interval, that is $G=F+c$ for some constant c.

5.23 Notation: We write

$$\int f=F+c\ ,\text{ or }\int F(x)\,dx=F(x)+c$$

when F is an antiderivative of f on an interval, so that the antiderivatives of f on the interval are the functions of the form $G=F+c$ for some constant c.

5.24 Example: Find $\displaystyle\int_0^{\sqrt{3}}\frac{dx}{1+x^2}$.

Solution: We have $\displaystyle\int\frac{dx}{1+x^2}=\tan^{-1}x+c$, since $\displaystyle\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$, and so by Part 2 of the Fundamental Theorem of Calculus, we have

$$\int_0^{\sqrt{3}}\frac{dx}{1+x^2}=\Big[\tan^{-1}x\Big]_0^{\sqrt{3}}=\tan^{-1}\sqrt{3}-\tan^{-1}0=\tfrac{\pi}{3}\,.$$

Chapter 6. Methods of Integration

Basic Integrals

6.1 Note: We have the following list of **Basic Integrals**

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + c\ ,\ \text{for } p \neq -1 \qquad \int \sec^2 x\ dx = \tan x + c$$

$$\int \frac{dx}{x} = \ln|x| + c \qquad \int \sec x \tan x\ dx = \sec x + c$$

$$\int e^x\,dx = e^x + c \qquad \int \tan x\ dx = \ln|\sec x| + c$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c \qquad \int \sec x\ dx = \ln|\sec x + \tan x| + c$$

$$\int \ln x\ dx = x\ln x - x + c \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x + c$$

$$\int \sin x\ dx = -\cos x + c \qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$$

$$\int \cos x\ dx = \sin x + c \qquad \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$$

Proof: Each of these equalities is easy to verify by taking the derivative of the right side. For example, we have $\displaystyle\int \ln x\ dx = x\ln x - x + c$ since $\displaystyle\frac{d}{dx}(x\ln x - x) = 1\cdot\ln x + x\cdot\tfrac{1}{x} - 1 = \ln x$, and we have $\displaystyle\int \tan x\ dx = \ln|\sec x| + c$ since $\displaystyle\frac{d}{dx}(\ln|\sec x|) = \frac{\sec x\tan x}{\sec x} = \tan x$, and we have $\displaystyle\int \sec x\ dx = \ln|\sec x + \tan x| + c$ since $\displaystyle\frac{d}{dx}(\ln|\sec x + \tan x|) = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x} = \sec x$.

6.2 Example: Find $\displaystyle\int_1^4 \frac{x^2-5}{\sqrt{x}}\,dx$.

Solution: By the Fundamental Theorem of Calculus and Linearity, we have

$$\int_1^4 \frac{x^2-5}{\sqrt{x}}\,dx = \int_1^4 x^{3/2} - 5x^{-1/2}\,dx = \Big[\tfrac{2}{5}x^{5/2} - 10x^{1/2}\Big]_1^4 = \big(\tfrac{64}{5} - 20\big) - \big(\tfrac{2}{5} - 10\big) = \tfrac{12}{5}\,.$$

Substitution

6.3 Theorem: *(Substitution, or Change of Variables) Let $u = g(x)$ be differentiable on an interval and let $f(u)$ be continuous on the range of $g(x)$. Then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

and

$$\int_{x=a}^{b} f(g(x))g'(x)\,dx = \int_{u=g(a)}^{g(b)} f(u)\,du\,.$$

Proof: Let $F(u)$ be an antiderivative of $f(u)$ so $F'(u) = f(u)$ and $\int f(u)\,du = F(u) + c$. Then from the Chain Rule, we have $\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$, and so

$$\int f(g(x))g'(x)\,dx = F(g(x)) + c = F(u) + c = \int f(u)\,du$$

and

$$\begin{aligned}\int_{x=a}^{b} f(g(x))g'(x)\,dx &= \Big[F(g(x))\Big]_{x=a}^{b} = F(g(b)) - F(g(a)) \\ &= \Big[F(u)\Big]_{u=g(a)}^{g(b)} = \int_{u=g(a)}^{g(b)} f(u)\,du\,.\end{aligned}$$

6.4 Notation: For $u = g(x)$ we write $du = g'(x)\,dx$. More generally, for $f(u) = g(x)$ we write $f'(u)\,du = g'(x)\,dx$. This notation makes the above theorem easy to remember and to apply.

6.5 Example: Find $\int \sqrt{2x+3}\,dx$.

Solution: Make the substitution $u = 2x + 3$ so $du = 2dx$. Then

$$\int \sqrt{2x+3}\,dx = \int \tfrac{1}{2}u^{1/2}\,du = \tfrac{1}{3}\,u^{3/2} + c = \tfrac{1}{3}(2x+3)^{3/2} + c\,.$$

(In applying the Substitution Rule, we used $u = g(x) = 2x + 3$ and $f(u) = \sqrt{u} = u^{1/2}$, but the notation $du = g'(x)\,dx$ allows us to avoid explicit mention of the function $f(u)$ in our solution).

6.6 Example: Find $\int x\,e^{x^2}\,dx$.

Solution: Make the substitution $u = x^2$ so $du = 2x\,dx$. Then

$$\int x\,e^{x^2}\,dx = \int \tfrac{1}{2}e^u\,du = \tfrac{1}{2}e^u + c = \tfrac{1}{2}e^{x^2} + c\,.$$

6.7 Example: Find $\displaystyle\int \frac{\ln x}{x}\,dx$.

Solution: Let $u = \ln x$ so $du = \dfrac{1}{x}\,dx$. Then

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \tfrac{1}{2}u^2 + c = \tfrac{1}{2}(\ln x)^2 + c\,.$$

6.8 Example: Find $\displaystyle\int \tan x\;dx$.

Solution: We have $\tan x = \dfrac{\sin x}{\cos x}$. Let $u = \cos x$ so $du = -\sin x\;dx$. Then

$$\int \tan x\;dx = \int \frac{\sin x\;dx}{\cos x} = \int \frac{-du}{u} = -\ln|u| + c = -\ln|\cos x| + c = \ln|\sec x| + c\,.$$

6.9 Example: Find $\displaystyle\int \frac{dx}{x+\sqrt{x}}$.

Solution: Let $u = \sqrt{x}$ so $u^2 = x$ and $2u\,du = dx$. Then

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2u\,du}{u^2+u} = \int \frac{2\,du}{u+1}\,.$$

Now let $v = u+1$ do $dv = du$. Then

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2\,du}{u+1} = \int \frac{2}{v}\,dv = 2\ln|v| + c = 2\ln|u+1| + c = 2\ln(\sqrt{x}+1) + c\,.$$

6.10 Example: Find $\displaystyle\int_0^2 \frac{x\,dx}{\sqrt{2x^2+1}}$.

Solution: Let $u = 2x^2+1$ so $du = 4x\,dx$. Then

$$\int_{x=0}^{2} \frac{x\,dx}{\sqrt{2x^2+1}} = \int_{u=1}^{9} \frac{\frac{1}{4}\,du}{\sqrt{u}} = \int_1^9 \tfrac{1}{4}\,u^{-1/2}\,du = \Big[\tfrac{1}{2}\,u^{1/2}\Big]_1^9 = \tfrac{3}{2} - \tfrac{1}{2} = 1\,.$$

6.11 Example: Find $\displaystyle\int_0^1 \frac{dx}{1+3x^2}$.

Solution: Let $u = \sqrt{3}\,x$ so $du = \sqrt{3}\,dx$. Then

$$\int_0^1 \frac{dx}{1+3x^2} = \int_0^{\sqrt{3}} \frac{\frac{1}{\sqrt{3}}\,du}{1+u^2} = \Big[\tfrac{1}{\sqrt{3}}\tan^{-1}u\Big]_0^{\sqrt{3}} = \tfrac{1}{\sqrt{3}}\,\tfrac{\pi}{3} = \tfrac{\pi}{3\sqrt{3}}\,.$$

Integration by Parts

6.12 Theorem: *(Integration by Parts) Let $f(x)$ and $g(x)$ be differentiable in an interval. Then*

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

so

$$\int_{x=a}^{b} f(x)g'(x)\,dx = \left[f(x)g(x) - \int g(x)f'(x)\,dx\right]_{x=a}^{b}.$$

Proof: By the Product Rule, we have $\dfrac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$ and so

$$\int f'(x)g(x) + f(x)g'(x)\ dx = f(x)g(x) + c\,,$$

which can be rewritten as

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx\,.$$

(We do not need to include the arbitrary constant c since there is now an integral on both sides of the equation).

6.13 Notation: If we write $u = f(x)$, $du = f'(x)\,dx$, $v = g(x)$ and $dv = g'(x)\,dx$, then the top formula in the above theorem becomes

$$\int u\,dv = uv - \int v\,du\,.$$

6.14 Note: To find the integral of a polynomial multiplied by an exponential function or a trigonometric function, try Integrating by parts with u equal to the polynomial (you may need to integrate by parts repeatedly if the polynomial is of high degree).

To integrate a polynomial (or an algebraic) function times a logarithmic or inverse trigonometric function, try integrating by parts letting u be the logarithmic or inverse trigonometric function.

To integrate an exponential function times a sine or cosine function, try integrating by parts twice, letting u be the exponential function both times.

6.15 Example: Find $\displaystyle\int x\sin x\ dx$.

Solution: Integrate by parts using $u = x$, $du = dx$, $v = -\cos x$ and $dv = \sin x\ dx$ to get

$$\int x\sin x\ dx = \int u\,dv = uv - \int v\,du = -x\cos x + \int \cos x\ dx = -x\cos x + \sin x + c\,.$$

6.16 Example: Find $\displaystyle\int (x^2+1)e^{2x}\,dx$.

Solution: Integrate by parts using $u = x^2+1$, $du = 2x\,dx$, $v = \frac{1}{2}e^{2x}$ and $dv = e^{2x}\,dx$ to get

$$\int (x^2+1)e^{2x}\,dx = \int u\,dv = uv - \int v\,du = \tfrac{1}{2}(x^2+1)e^{2x} - \int x\,e^{2x}\,dx\,.$$

To find $\displaystyle\int x\,e^{2x}\,dx$ we integrate by parts again, this time using $u_2 = x$, $du_2 = dx$, $v_2 = \frac{1}{2}e^{2x}$ and $dv_2 = e^{2x}\,dx$ to get

$$\begin{aligned}\int (x^2+1)e^{2x}\,dx &= \tfrac{1}{2}(x^2+1)e^{2x} - \int x\,e^{2x}\,dx \\ &= \tfrac{1}{2}(x^2+1)e^{2x} - \left(\tfrac{1}{2}\,xe^{2x} - \int \tfrac{1}{2}e^{2x}\,dx\right) \\ &= \tfrac{1}{2}(x^2+1)e^{2x} - \left(\tfrac{1}{2}\,xe^{2x} - \tfrac{1}{4}e^{2x}\right) + c \\ &= \tfrac{1}{4}\left(2x^2-2x+3\right)e^{2x} + c\end{aligned}$$

6.17 Example: Find $\displaystyle\int \ln x\;dx$.

Solution: Integrate by parts using $u = \ln x$, $du = \dfrac{1}{x}\,dx$, $v = x$ and $dv = dx$ to get

$$\int \ln x\;dx = x\ln x - \int 1\,dx = x\ln x - x + c\,.$$

6.18 Example: Find $\displaystyle\int_1^4 \sqrt{x}\;\ln x\;dx$.

Solution: Integrate by parts using $u = \ln x$, $du = \dfrac{1}{x}\,dx$, $v = \frac{2}{3}x^{3/2}$ and $dv = x^{1/2}\,dx$ to get

$$\begin{aligned}\int_1^4 \sqrt{x}\;\ln x\;dx &= \left[\tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{1/2}\,dx\right]_1^4 = \left[\tfrac{2}{3}x^{3/2}\ln x - \tfrac{4}{9}x^{3/2}\right]_1^4 \\ &= \left(\tfrac{16}{3}\ln 4 - \tfrac{32}{9}\right) - \left(\tfrac{2}{3}\ln 1 - \tfrac{4}{9}\right) = \tfrac{16}{3}\ln 4 - \tfrac{28}{9}\,.\end{aligned}$$

6.19 Example: Find $\displaystyle\int e^x \sin x\;dx$

Solution: Write $I = \displaystyle\int e^x\sin x\,dx$. Integrate by parts twice, first using $u_1 = e^x$, $du = e^x\,dx$, $v = -\cos x$ and $dv = \sin x\;dx$, and next using $u_2 = e^x$, $du_2 = e^x\,dx$, $v_2 = \sin x$ and $dv_2 = \cos x\;dx$ to get

$$\begin{aligned}I &= -e^x\cos x + \int e^x\cos x\;dx \\ &= -e^x\cos x + \left(e^x\sin x - \int e^x\sin x\;dx\right)\,. \\ &= -e^x\cos x + e^x\sin x - I\end{aligned}$$

Thus $2I = -e^x\cos x + e^x\sin x + c$ and so $I = \frac{1}{2}(\sin x - \cos x)e^x + d$.

6.20 Example: Let $n \geq 2$ be an integer. Find a formula for $\displaystyle\int \sin^n x\ dx$ in terms of $\displaystyle\int \sin^{n-2} x\ dx$, and hence find $\displaystyle\int \sin^2 x\ dx$ and $\displaystyle\int \sin^4 x\ dx$.

Solution: Let $I = \displaystyle\int \sin^n x\ dx = \int \sin^{n-1} x \sin x\ dx$. Integrate by parts using $u = \sin^{n-1} x$, $du = (n-1)\sin^{n-2} x \cos x\, dx$, $v = -\cos x$ and $dv = \sin x\, dx$ to get

$$\begin{aligned} I &= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x \cos^2 x\ dx \\ &= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x(1-\sin^2 x)\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)I\,. \end{aligned}$$

Add $(n-1)I$ to both sides to get $nI = -\sin^{n-1} x \cos x + (n-1)\displaystyle\int \sin^{n-2} x\ dx$, that is

$$\int \sin^n x\ dx = -\tfrac{1}{n}\sin^{n-1} x \cos x + \tfrac{n-1}{n}\int \sin^{n-2} x\ dx\,.$$

In particular, when $n=2$ we get

$$\int \sin^2 x\ dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}\int 1\, dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x + c$$

and when $n=4$ we get

$$\int \sin^4 x\ dx = -\tfrac{1}{4}\sin^3 x \cos x + \tfrac{3}{4}\int \sin^2 x\ dx = -\tfrac{1}{4}\sin^3 x \cos x - \tfrac{3}{8}\sin x \cos x + \tfrac{3}{8}x + c\,.$$

6.21 Example: Let $n \geq 2$ be an integer. Find a formula for $\displaystyle\int \sec^n x\ dx$ in terms of $\displaystyle\int \sec^{n-2} x\ dx$, and hence find $\displaystyle\int \sec^3 x\ dx$.

Solution: Let $I = \displaystyle\int \sec^n x\ dx = \int \sec^{n-2} x \sec^2 x\ dx$. Using Integrate by Parts with $u = \sec^{n-2} x$, $du = (n-2)\sec^{n-2} x \tan x\, dx$, $v = \tan x$ and $dv = \tan x\, dx$, we obtain

$$\begin{aligned} I &= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x\ dx \\ &= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x(\sec^2 x - 1)\ dx \\ &= \sec^{n-2} x \tan x - (n-2)I + (n-2)\int \sec^{n-2} x\ dx \end{aligned}$$

Add $(n-2)I$ to both sides to get $(n-1)I = \sec^{n-2} x \tan x + (n-2)\displaystyle\int \sec^{n-2} x\ dx$, that is

$$\int \sec^n x\ dx = \tfrac{1}{n-1}\sec^{n-2} x \tan x + \tfrac{n-2}{n-1}\int \sec^{n-2} x\ dx\,.$$

In particular, when $n=3$ we get

$$\int \sec^3 x\ dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\int \sec x\ dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln\left|\sec x + \tan x\right| + c$$

6.22 Note: To find $\int f(\sin x)\cos^{2n+1} x\ dx$, write $\cos^{2n+1} x = (1-\sin^2 x)^n \cos x$ then try the substitution $u = \sin x$, $du = \cos x\,dx$.

To find $\int f(\cos x)\sin^{2n+1} x\ dx$, write $\sin^{2n+1} x = (1-\cos^2 x)^n \sin x$ then try the substitution $u = \cos x$, $du = -\sin x\ dx$.

To find $\int \sin^{2m} x\cos^{2n} x\ dx$, try using the trigonometric identities $\sin^2\theta = \frac{1}{2}-\frac{1}{2}\cos 2\theta$ and $\cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta$. Alternatively, write $\cos^{2n} x = (1-\sin^2 x)^n$ and use the formula from Example 2.20.

To find $\int f(\tan x)\sec^{2n+2} x\ dx$, write $\sec^{2n+2} x = (1+\tan^2 x)^n \sec^2 x\,dx$ and try the substitution $u = \tan x$, $du = \sec^2 x\,dx$.

To find $\int f(\sec x)\tan^{2n+1} x\ dx$, write $\tan^{2n+1} x = \dfrac{(\sec^2 x - 1)^n}{\sec x}\sec x\tan x\ dx$ and try the substitution $u = \sec x$, $du = \sec x\tan x\ dx$.

To find $\int \sec^{2n+1} x\tan^{2n} x\ dx$, write $\tan^{2n} x = (\sec^2 x - 1)^n$ and use the formula from Example 2.21.

6.23 Example: Find $\displaystyle\int_0^{\pi/3} \frac{\sin^3 x}{\cos^2 x}\,dx$.

Solution: Make the substitution $u = \cos x$ so $du = -\sin x\,dx$. Then

$$\int_0^{\pi/3} \frac{\sin^3 x}{\cos^2 x}\,dx = \int_0^{\pi/3} \frac{(1-\cos^2 x)\sin x\,dx}{\cos^2 x} = \int_1^{1/2} -\frac{(1-u^2)\,du}{u^2} = \int_1^{1/2} -\frac{1}{u^2}+1\ du$$
$$= \Big[\tfrac{1}{u}+u\Big]_1^{1/2} = \big(2+\tfrac{1}{2}\big)-(1+1) = \tfrac{1}{2}\,.$$

6.24 Example: Find $\int \sin^6 x\ dx$.

Solution: We could use the method of example 2.20, but we choose instead to use the half-angle formulas. We have

$$\int_0^{\pi/4} \sin^6 x\ dx = \int_0^{\pi/4} \big(\tfrac{1}{2}-\tfrac{1}{2}\cos 2x\big)^3\,dx = \int_0^{\pi/4} \tfrac{1}{8}-\tfrac{3}{8}\cos 2x+\tfrac{3}{8}\cos^2 2x-\tfrac{1}{8}\cos^3 2x\ dx$$
$$= \int_0^{\pi/4} \tfrac{1}{8}-\tfrac{3}{8}\cos 2x+\tfrac{3}{8}\big(\tfrac{1}{2}+\tfrac{1}{2}\cos 4x\big)-\tfrac{1}{8}\big(1-\sin^2 2x\big)\cos 2x\ dx$$
$$= \int_0^{\pi/4} \tfrac{5}{16}-\tfrac{1}{2}\cos 2x+\tfrac{3}{16}\cos 4x+\tfrac{1}{8}\sin^2 2x\cos 2x\ dx$$
$$= \Big[\tfrac{5}{15}x-\tfrac{1}{4}\sin 2x+\tfrac{3}{64}\sin 4x+\tfrac{1}{48}\sin^3 2x\Big]_0^{\pi/4}$$
$$= \tfrac{5\pi}{64}-\tfrac{1}{4}+\tfrac{1}{48} = \tfrac{5\pi}{64}-\tfrac{11}{48}\,.$$

6.25 Example: Find $\displaystyle\int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x + 1}}\, dx$.

Solution: Make the substitution $u = \tan x$ so $du = \sec^2 x\, dx$. Then

$$\int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x + 1}}\, dx = \int_0^{\pi/4} \frac{(\tan^2 x + 1)\sec^2 x\, dx}{\sqrt{\tan x + 1}} = \int_0^1 \frac{(u^2+1)\, du}{\sqrt{u+1}}$$

Now make the substitution $v = u + 1$ so $u = v - 1$ and $du = dv$. Then

$$\begin{aligned}\int_0^1 \frac{u^2+1}{\sqrt{u+1}}\, du &= \int_1^2 \frac{(v-1)^2+1}{\sqrt{v}}\, dv = \int_1^2 v^{3/2} - 2\, v^{1/2} + 2\, v^{-1/2}\ dv \\ &= \Big[\tfrac{2}{5}\, v^{5/2} - \tfrac{4}{3}\, v^{3/2} + 4\, v^{1/2}\Big]_1^2 = \big(\tfrac{2\cdot 4\sqrt{2}}{5} - \tfrac{4\cdot 2\sqrt{2}}{3} + 4\sqrt{2}\big) - \big(\tfrac{2}{5} - \tfrac{4}{3} + 4\big) \\ &= \tfrac{(24-40+60)\sqrt{2}}{15)} - \tfrac{6-20+60}{15} = \tfrac{44\sqrt{2}-46}{15}\,.\end{aligned}$$

6.26 Example: Find $\displaystyle\int_0^{\pi/4} \tan^4 x\ dx$.

Solution: Note first that

$$\tan^4 x = \tan^2 x(\sec^2 x - 1) = \tan^2 x \sec^2 x - \tan^2 x = \tan^2 x \sec^2 x - \sec^2 x + 1\,.$$

To find $\displaystyle\int \tan^2 x \sec^2 x\ dx$, make the substitution $u = \tan\theta$, $du = \sec^2\theta\, d\theta$ to get

$$\int \tan^2 x \sec^2 x\ dx = \int u^2\, du = \tfrac{1}{3}u^3 + c = \tfrac{1}{3}\tan^3 x + c\,.$$

Thus we have

$$\begin{aligned}\int_0^{\pi/4} \tan^4 x\, dx &= \int_0^{\pi/4} \tan^2 x \sec^2 x - \sec^2 x + 1 \\ &= \Big[\tfrac{1}{3}\ \tan^3 x - \tan x + x\Big]_0^{\pi/4} = \tfrac{1}{3} - 1 + \tfrac{\pi}{4} = \tfrac{\pi}{4} - \tfrac{2}{3}\,.\end{aligned}$$

6.27 Note: To find $\displaystyle\int \sin(ax)\sin(bx)\, dx$, $\displaystyle\int \cos(ax)\cos(bx)\, dx$, or $\displaystyle\int \sin(ax)\cos(bx)\, dx$, use one of the identities

$$\begin{aligned}\cos(A-B) - \cos(A+B) &= 2\sin A \sin B \\ \cos(A-B) + \cos(A+B) &= 2\cos A \cos B \\ \sin(A-B) + \sin(A+B) &= 2\sin A \cos B\,.\end{aligned}$$

6.28 Example: Find $\displaystyle\int_0^{\pi/6} \cos 3x \cos 2x\, dx$.

Solution: Since $2\cos 3x \cos 2x = \cos(3x - 2x) + \cos(3x + 2x) = \cos x + \cos 5x$, we have

$$\int_0^{\pi/6} \cos 2x \cos 3x\, dx = \int_0^{\pi/6} \tfrac{1}{2}(\cos x + \cos 5x)\ dx = \Big[\tfrac{1}{2}\sin x + \tfrac{1}{10}\sin 5x\Big]_0^{\pi/6} = \tfrac{1}{4} + \tfrac{1}{20} = \tfrac{3}{10}\,.$$

Inverse Trigonometric Substitution

6.29 Note: To solve an integral involving $\sqrt{a^2+b^2(x+c)^2}$ or $1/(a^2+b^2(x+c)^2)$, try the substitution $\theta=\tan^{-1}\frac{b(x+c)}{a}$ so that $a\tan\theta=b(x+c)$, $a\sec\theta=\sqrt{a^2+b^2(x+c)^2}$ and $a\sec^2\theta\,d\theta=b\,dx$.

For an integral involving $\sqrt{a^2-b^2(x+c)^2}$, try the substitution $\theta=\sin^{-1}\frac{b(x+c)}{a}$ so that $a\sin\theta=b(x+c)$, $a\cos\theta=\sqrt{a^2-b^2(x+c)^2}$ and $a\cos\theta\,d\theta=b\,dx$.

For an integral involving $\sqrt{b^2(x+c)^2-a^2}$, try the substitution $\theta=\sec^{-1}\frac{b(x+c)}{a}$ so that $a\sec\theta=b(x+c)$, $a\tan\theta=\sqrt{b^2(x+c)^2-a^2}$ and $a\sec\theta\tan\theta\,d\theta=b\,dx$.

6.30 Example: Find $\displaystyle\int_0^1\frac{dx}{(4-3x^2)^{3/2}}$.

Solution: Let $2\sin\theta=\sqrt{3}\,x$ so $2\cos\theta=\sqrt{4-3x^2}$ and $2\cos\theta\,d\theta=\sqrt{3}\,dx$. Then

$$\int_0^1\frac{dx}{(4-3x^2)^{3/2}}=\int_0^{\pi/3}\frac{\frac{2}{\sqrt{3}}\cos\theta\,d\theta}{(2\cos\theta)^3}=\int_0^{\pi/3}\tfrac{1}{4\sqrt{3}}\sec^2\theta\,d\theta=\Big[\tfrac{1}{4\sqrt{3}}\tan\theta\Big]_0^{\pi/3}=\tfrac{1}{4}\,.$$

6.31 Example: Find $\displaystyle\int_1^{\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}}$.

Solution: Let $\sqrt{3}\tan\theta=x$ so $\sqrt{3}\sec\theta=\sqrt{x^2+3}$ and $\sqrt{3}\sec^2\theta\,d\theta=dx$, and also let $u=\sin\theta$ so $du=\cos\theta\,d\theta$. Then

$$\begin{aligned}\int_1^{\sqrt{3}}\frac{dx}{x^2\sqrt{x^2+3}}&=\int_{\pi/6}^{\pi/4}\frac{\sqrt{3}\sec^2\theta\,d\theta}{3\tan^2\theta\sqrt{3}\sec\theta}=\int_{\pi/6}^{\pi/4}\frac{1}{3}\frac{\sec\theta}{\tan^2\theta}\,d\theta=\int_{\pi/6}^{\pi/4}\frac{1}{3}\frac{\cos\theta\,d\theta}{\sin^2\theta}\\&=\int_{1/2}^{1/\sqrt{2}}\frac{1}{3\,u^2}\,du=\Big[-\frac{1}{3u}\Big]_{1/2}^{1/\sqrt{2}}=-\tfrac{\sqrt{2}}{3}+\tfrac{2}{3}=\tfrac{2-\sqrt{2}}{3}\,.\end{aligned}$$

6.32 Example: Find $\displaystyle\int_2^4\frac{\sqrt{x^2-4}}{x^2}\,dx$.

Solution: Let $2\sec\theta=x$ so $2\tan\theta=\sqrt{x^2-4}$ and $2\sec\theta\tan\theta\,d\theta=dx$. Then

$$\begin{aligned}\int_2^4\frac{\sqrt{x^2-4}}{x^2}\,dx&=\int_0^{\pi/3}\frac{\tan^2\theta\sec\theta\,d\theta}{\sec^2\theta}=\int_0^{\pi/3}\frac{\tan^2\theta}{\sec\theta}\,d\theta=\int_0^{\pi/3}\frac{\sec^2\theta-1}{\sec\theta}\,d\theta\\&=\int_0^{\pi/3}\sec\theta-\cos\theta\;d\theta=\Big[\ln|\sec\theta+\tan\theta|-\sin\theta\Big]_0^{\pi/3}=\ln(2+\sqrt{3})-\tfrac{\sqrt{3}}{2}\,.\end{aligned}$$

6.33 Example: Find $\displaystyle\int_2^3(4x-x^2)^{3/2}\,dx$.

Solution: Let $2\sin\theta=x-2$ so $2\cos\theta=\sqrt{4x-x^2}$ and $2\cos\theta\,d\theta=dx$. Then

$$\begin{aligned}\int_2^3(4x-x^2)^{3/2}\,dx&=\int_0^{\pi/6}16\cos^4\theta\,d\theta=\int_0^{\pi/6}4\,(1+\cos2\theta)^2\,d\theta\\&=\int 4+8\cos2\theta+4\cos^22\theta\;d\theta=\int 4+8\cos2\theta+2+2\cos4\theta\;d\theta\\&=\Big[6\theta+4\sin2\theta+\tfrac{1}{2}\sin4\theta\Big]_0^{\pi/6}=\pi+2\sqrt{3}+\tfrac{\sqrt{3}}{4}=\pi+\tfrac{9\sqrt{3}}{4}\,.\end{aligned}$$

Partial Fractions

6.34 Note: We can find the integral of a rational function $\dfrac{f(x)}{g(x)}$ as follows:

Step 1: use long division to find polynomials $q(x)$ and $r(x)$ with $\deg r(x) < \deg g(x)$ such that $f(x) = g(x)q(x) + r(x)$ for all x, and note that $\dfrac{f(x)}{g(x)} = q(x) + \dfrac{r(x)}{g(x)}$ so

$$\int \frac{f(x)}{g(x)}\,dx = \int q(x) + \frac{r(x)}{g(x)}\,dx\,.$$

(If $\deg f(x) < \deg g(x)$ then $q(x) = 0$ and $r(x) = f(x)$).

Step 2: factor $g(x)$ into linear and irreducible quadratic factors.

Step 3: write $\dfrac{r(x)}{g(x)}$ as a sum of terms so that for each linear factor $(ax+b)^k$ we have the k terms

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

and for each irreducible quadratic factor $(ax^2+bx+c)^k$ we have the k terms

$$\frac{B_1x+C_1}{(ax^2+bx+c)} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_kx+C_k}{(ax^2+bx+c)^k}\,.$$

Writing $\dfrac{r(x)}{g(x)}$ in this form is called splitting $\dfrac{r(x)}{g(x)}$ into its **partial fractions** decomposition.

Step 4: solve the integral.

6.35 Example: If $g(x) = x(x-1)^3(x^2+2x+3)^2$ then in step 3 we would write

$$\frac{r(x)}{g(x)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} + \frac{Ex+F}{x^2+2x+3} + \frac{Gx+H}{(x^2+2x+3)^2}\,.$$

and then solve for the various constants.

6.36 Example: Find $\displaystyle\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx$.

Solution: In order to get $\dfrac{x-7}{(x-1)^2(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2}$ we need

$$A(x-1)(x+2) + B(x+2) + C(x-1)^2 = x-7\,.$$

Equating coefficients gives $A+C=0$, $A+B-2C=1$ and $-2A+2B+C=-7$. Solving these three equations gives $A=1$, $B=-2$ and $C=-1$, and so we have

$$\begin{aligned}\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx &= \int_2^3 \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} \\ &= \int_2^3 \frac{1}{x-1} - \frac{2}{(x-1)^2} - \frac{1}{x+2}\,dx = \Big[\ln(x-1) + \tfrac{2}{x-1} - \ln(x+2)\Big]_2^3 \\ &= (\ln 2 + 1 - \ln 5) - (2 - \ln 4) = \ln\frac{8}{5} - 1\,.\end{aligned}$$

6.37 Example: Find $\displaystyle\int_1^{\sqrt{3}} \frac{x^4 - x^3 + 1}{x^3 + x}\,dx$.

Solution: Use long division of polynomials to show that $\dfrac{x^4 - x^3 + 1}{x^3 + x} = x - 1 + \dfrac{-x^2 + x + 1}{x^3 + x}$. Next, note that to get $\dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1} = \dfrac{-x^2 + x + 1}{x^3 + x}$ we need $A(x^2 + 1) + (Bx + C)(x) = -x^2 + x + 1$. Equating coefficients gives $A + B = -1$, $C = 1$ and $A = 1$. Solving these three equations gives $A = 1$, $B = -2$ and $C = 1$. Thus

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{x^4 - x^3 + 1}{x^3 + x}\,dx &= \int_1^{\sqrt{3}} x - 1 + \frac{1}{x} - \frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1}\,dx \\
&= \Big[\tfrac{1}{2}\,x^2 - x + \ln x - \ln(x^2 + 1) + \tan^{-1} x\Big]_1^{\sqrt{3}} \\
&= \left(\tfrac{3}{2} - \sqrt{3} + \ln\sqrt{3} - \ln 4 + \tfrac{\pi}{3}\right) - \left(\tfrac{1}{2} - 1 - \ln 2 + \tfrac{\pi}{4}\right) \\
&= 2 - \sqrt{3} + \ln\tfrac{\sqrt{3}}{2} + \tfrac{\pi}{12}\,.
\end{aligned}$$

6.38 Example: Find $I = \displaystyle\int_1^2 \frac{x^5 + x^4 - 2x^3 - 2x^2 - 5x - 25}{x^2(x^2 - 2x + 5)^2}\,dx$.

Solution: To get

$$\frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 - 2x + 5} + \frac{Ex + F}{(x^2 - 2x + 5)^2} = \frac{x^5 + x^4 - 2x^3 - 2x^2 - 5x - 25}{x^2(x^2 - 2x + 5)^2}$$

we need $Ax(x^2 - 2x + 5)^2 + B(x^2 - 2x + 5)^2 + (Cx + D)(x^2)(x^2 - 2x + 5) + (Ex + F)(x^2) = x^5 + x^4 - 2x^3 - 2x^2 - 5x - 25$. Expanding the left hand side then equating coefficients gives the 5 equations

$$\begin{aligned}
&A + C = 1\ ,\ -4A + B - 2C + D = 1\ ,\ 14A - 4B + 5C - 2D + E = -2 \\
&-20A + 14B + 5D + F = -2\ ,\ 25A - 20B = -5\ ,\ 25B = -25
\end{aligned}$$

Solving these equations gives $A = -1$, $B = -1$, $C = 2$, $D = 2$, $E = 2$ and $F = -18$, so

$$\begin{aligned}
I &= \int_1^2 -\frac{1}{x} - \frac{1}{x^2} + \frac{2x + 2}{x^2 - 2x + 5} + \frac{2x - 18}{(x^2 - 2x + 5)^2}\,dx \\
&= \int_1^2 -\frac{1}{x} - \frac{1}{x^2} + \frac{2x - 2 + 4}{x^2 - 2x + 5} + \frac{2x - 2 - 16}{(x^2 - 2x + 5)^2}\,dx \\
&= \int_1^2 -\frac{1}{x} - \frac{1}{x^2} + \frac{2x - 2}{x^2 - 2x + 5} + \frac{4}{x^2 - 2x + 5} + \frac{2x - 2}{(x^2 - 2x + 5)^2} - \frac{16}{(x^2 - 2x + 5)^2}\,dx
\end{aligned}$$

We have $\displaystyle\int \frac{1}{x}\,dx = \ln x + c$ and $\displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x} + c$. Make the substitution $u = x^2 - 2x + 5$, $du = (2x - 2)\,dx$ to get

$$\int \frac{(2x - 2)\,dx}{x^2 - 2x + 5} = \int \frac{du}{u} = \ln u + c = \ln(x^2 - 2x + 5) + c$$

and

$$\int \frac{(2x - 2)\,dx}{(x^2 - 2x + 5)^2} = \int \frac{du}{u^2} = \frac{-1}{u} + c = \frac{-1}{x^2 - 2x + 5} + c.$$

Make the substitution $2\tan\theta = x - 1$, $2\sec\theta = \sqrt{x^2 - 2x + 5}$, $2\sec^2\theta\,d\theta = dx$ to get

$$\int \frac{4\,dx}{x^2 - 2x + 5} = \int \frac{4 \cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^2} = \int 2\,d\theta = 2\theta + c = 2\tan^{-1}\left(\tfrac{x-1}{2}\right) + c$$

and

$$\int \frac{16\,dx}{(x^2-2x+5)^2} = \int \frac{16\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^4}\,d\theta = \int \frac{2\,d\theta}{\sec^2\theta} = \int 2\cos^2\theta\,d\theta = \int 1+\cos 2\theta\;d\theta$$
$$= \theta + \tfrac{1}{2}\,\sin 2\theta + c = \theta + \sin\theta\cos\theta + c = \tan^{-1}\left(\tfrac{x-1}{2}\right) + \tfrac{2(x-1)}{x^2-2x+5} + c\,.$$

Thus we have

$$\begin{aligned} I &= \Bigg[-\ln x + \frac{1}{x} + \ln(x^2-2x+5) + 2\tan^{-1}\frac{x-1}{2} \\ &\qquad - \frac{1}{x^2-2x+5} - \tan^{-1}\frac{x-1}{2} - \frac{2(x-1)}{x^2-2x+5}\Bigg]_1^2 \\ &= \left[\ln\frac{x^2-2x+5}{x} + \frac{1}{x} - \frac{2x-1}{x^2-2x+5} + \tan^{-1}\frac{x-1}{2}\right]_1^2 \\ &= \left(\ln\tfrac{5}{2} + \tfrac{1}{2} - \tfrac{3}{5} + \tan^{-1}\tfrac{1}{2}\right) - \left(\ln 4 + 1 - \tfrac{1}{4}\right) \\ &= \ln\tfrac{5}{8} - \tfrac{17}{20} + \tan^{-1}\tfrac{1}{2}\,. \end{aligned}$$

6.39 Example: Find $\displaystyle\int \frac{\sec^3 x\,dx}{\sec x - 1}$.

Solution: Multiply the numerator and denominator by $\sec x + 1$ to get

$$\int \frac{\sec^3 x\,dx}{\sec x - 1} = \int \frac{\sec^3 x(\sec x + 1)}{(\sec^2 x - 1)}\,dx = \int \frac{\sec^4 x + \sec^3 x}{\tan^2 x}\,dx = \int \frac{\sec^4 x}{\tan^2 x}\,dx + \int \frac{\sec^3 x}{\tan^2 x}\,dx\,.$$

Make the substitution $u = \tan x$, $du = \sec^2 x\,dx$ to get

$$\begin{aligned} \int \frac{\sec^4 x}{\tan^2 x}\,dx &= \int \frac{(\tan^2 x + 1)\sec^2 x\,dx}{\tan^2 x} = \int \frac{u^2+1}{u^2}\,du \\ &= \int 1 + \frac{1}{u^2}\,du = u - \frac{1}{u} + c = \tan x - \cot x + c\,. \end{aligned}$$

Make the substitution $v = \sin x$, $dv = \cos x\,dx$ and integrate by parts to get

$$\begin{aligned} \int \frac{\sec^3 x}{\tan^2 x}\,dx &= \int \frac{dx}{\cos x \sin^2 x} = \int \frac{\cos x\,dx}{(1-\sin^2 x)\sin^2 x} = \int \frac{dv}{(1-v^2)\,v^2} \\ &= \int \frac{1}{1-v^2} + \frac{1}{v^2}\;dv = \int \frac{\frac{1}{2}}{1-v} + \frac{\frac{1}{2}}{1+v} + \frac{1}{v^2}\;dv \\ &= -\tfrac{1}{2}\ln|1-v| + \tfrac{1}{2}\ln|1+v| - \tfrac{1}{v} + c = \tfrac{1}{2}\ln\left|\tfrac{1+v}{1-v}\right| - \tfrac{1}{v} + c \\ &= \tfrac{1}{2}\ln\tfrac{1+\sin x}{1-\sin x} - \csc x + c = \tfrac{1}{2}\ln\tfrac{(1+\sin x)^2}{(\cos x)^2} - \csc x + c = \ln\left|\tfrac{1+\sin x}{\cos x}\right| - \csc x + c\,. \end{aligned}$$

Thus $\displaystyle\int \frac{\sec^3 x}{\sec x - 1}\,dx = \tan x - \cot x + \ln|\sec x + \tan x| - \csc x + c.$

Improper Integration

6.40 Definition: Suppose that $f:[a,b)\to\mathbf{R}$ is integrable on every closed interval contained in $[a,b)$. Then we define the **improper integral** of f on $[a,b)$ to be

$$\int_a^b f=\lim_{t\to b^-}\int_a^t f$$

provided the limit exists and, when the improper integral exists and is finite, we say that f is **improperly integrable** on $[a,b)$, (or that the improper integral of f on $[a,b)$ **converges**). In this definition we also allow the case that $b=\infty$, and then we have

$$\int_a^\infty f=\lim_{t\to\infty}\int_a^t f\,.$$

Similarly, if $f:(a,b]\to\mathbf{R}$ is integrable on every closed interval in $(a,b]$ then we define the **improper integral** of f on $(a,b]$ to be

$$\int_a^b f=\lim_{t\to a^+}\int_t^b f$$

provided the limit exists, and we say that f is **improperly integrable** on $(a,b]$ when the improper integral is finite. In this definition we also allow the case that $a=-\infty$. For a function $f:(a,b)\to\mathbf{R}$, which is integrable on every closed interval in (a,b), we choose a point $c\in(a,b)$, then we define the **improper integral** of f on (a,b) to be

$$\int_a^b f=\int_a^c f+\int_c^b f$$

provided that both of the improper integrals on the right exist and can be added, and we say that f is **improperly integrable** on (a,b) when both of the improper integrals on the right are finite. As an exercise, you should verify that the value of this integral does not depend on the choice of c.

6.41 Notation: For a function $F:(a,b)\to\mathbf{R}$ write

$$\Big[F(x)\Big]_{a^+}^{b^-}=\lim_{x\to b^-}F(x)-\lim_{x\to a^+}F(x)\,.$$

We use similar notation when $F:[a,b)\to\mathbf{R}$ and when $F:(a,b]\to\mathbf{R}$.

6.42 Note: Suppose that $f:(a,b)\to\mathbf{R}$ is integrable on every closed interval contained in (a,b) and that F is differentiable with $F'=f$ on (a,b). Then

$$\int_a^b f=\Big[F(x)\Big]_{a^+}^{b^-}\,.$$

A similar result holds for functions defined on half-open intervals $[a,b)$ and $(a,b]$.

Proof: Choose $c\in(a,b)$. By the Fundamental Theorem of Calculus we have

$$\begin{aligned}\int_a^b f=\int_a^c f+\int_c^b f&=\lim_{s\to a^+}\int_s^c f+\lim_{t\to b^-}\int_c^t f\\&=\lim_{s\to a^+}\big(F(c)-F(s)\big)+\lim_{t\to b^-}\big(F(t)-F(c)\big)\\&=\lim_{t\to b^-}F(t)-\lim_{s\to a^+}F(s)=\Big[F(x)\Big]_{a^+}^{b^-}\,.\end{aligned}$$

6.43 Example: Find $\displaystyle\int_0^1 \frac{dx}{x}$ and find $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$.

Solution: We have

$$\int_0^1 \frac{dx}{x} = \Big[\ln x\Big]_{0^+}^1 = 0 - (-\infty) = \infty$$

and

$$\int_0^1 \frac{dx}{\sqrt{x}} = \Big[2\sqrt{x}\Big]_{0^+}^1 = 2 - 0 = 2\,.$$

6.44 Example: Show that $\displaystyle\int_0^1 \frac{dx}{x^p}$ converges if and only if $p < 1$.

Solution: The case that $p = 1$ was dealt with in the previous example. If $p > 1$ so that $p - 1 > 0$ then we have

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}}\right]_{0^+}^1 = \left(-\tfrac{1}{p-1}\right) - \left(-\infty\right) = \infty$$

and if $p < 1$ so that $1 - p > 0$ then we have

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_{0^+}^1 = \left(\tfrac{1}{1-p}\right) - (0) = \tfrac{1}{1-p}\,.$$

6.45 Example: Show that $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converges if and only if $p > 1$.

Solution: When $p = 1$ we have

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{1}{x} = \Big[\ln x\Big]_1^\infty = \infty - 0 = \infty\,.$$

When $p > 1$ so that $p - 1 > 0$ we have

$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}}\right]_1^\infty = (0) - \left(-\tfrac{1}{p-1}\right) = \tfrac{1}{p-1}$$

and if $p < 1$ so that $1 - p > 0$ then we have

$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^\infty = (\infty) - \left(\tfrac{1}{1-p}\right) = \infty\,.$$

6.46 Example: Find $\displaystyle\int_0^\infty e^{-x}\,dx$.

Solution: We have
$$\int_0^\infty e^{-x}\,dx = \Big[-e^{-x}\Big]_0^\infty = 0-(-1) = 1\,.$$

6.47 Example: Find $\displaystyle\int_0^1 \ln x\,dx$.

Solution: We have
$$\int_0^1 \ln x\;dx = \Big[x\ln x - x\Big]_{0^+}^1 = (-1)-(0) = -1\,,$$
since l'Hôpital's Rule gives $\displaystyle\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$

6.48 Theorem: *(Comparison) Let f and g be integrable on closed subintervals of (a,b), and suppose that $0\le f(x)\le g(x)$ for all $x\in(a,b)$. If g is improperly integrable on (a,b) then so is f and then we have*
$$\int_a^b f \le \int_a^b g\,.$$
On the other hand, if $\displaystyle\int_a^b f$ diverges then $\displaystyle\int_a^b g$ diverges, too. A similar result holds for functions f and g defined on half-open intervals.

Proof: The proof is left as an exercise.

6.49 Example: Determine whether $\displaystyle\int_0^{\pi/2}\sqrt{\sec x}\,dx$ converges.

Solution: For $0\le x<\frac{\pi}{2}$ we have $\cos x\ge 1-\frac{2}{\pi}\,x$ so $\sec x\le \frac{1}{1-\frac{2}{\pi}\,x}$ hence $\sqrt{\sec x}\le\frac{1}{\sqrt{1-\frac{2}{\pi}\,x}}$. Let $u=1-\frac{2}{\pi}\,x$ so that $du=-\frac{2}{\pi}\,dx$. Then
$$\int_{x=0}^{\pi/2}\frac{1}{\sqrt{1-\frac{2}{\pi}\,x}}\,dx = \int_{u=1}^0 -\tfrac{\pi}{2}\,u^{-1/2} = \Big[-\pi\,u^{1/2}\Big]_1^0 = \pi$$
which is finite. It follows that $\displaystyle\int_0^{\pi/2}\sqrt{\sec x}\,dx$ converges, by comparison.

6.50 Example: Determine whether $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges.

Solution: For $0\le u$ we have $e^u\ge 1+u$, so for $0\le x$ we have $e^{x^2}\ge 1+x^2$, so $e^{-x^2}\le\dfrac{1}{1+x^2}$. Since
$$\int_0^\infty\frac{dx}{1+x^2} = \Big[\tan^{-1}x\Big]_0^\infty = \tfrac{\pi}{2}\,,$$
which is finite, we see that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges, by comparison.

6.51 Theorem: *(Estimation) Let f be integrable on closed subintervals of (a,b). If $|f|$ is improperly integrable on (a,b) then so is f, and then we have*

$$\left|\int_a^b f\right| \leq \int_a^b |f|\,.$$

A similar result holds for functions defined on half-open intervals.

Proof: The proof is left as an exercise.

6.52 Example: Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ converges.

Solution: We shall show that both of the integrals $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ and $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$ converge.

Since $\displaystyle\lim_{x\to 0^+} \frac{\sin x}{x} = 1$, the function f defined by $f(0)=1$ and $f(x) = \dfrac{\sin x}{x}$ for $x>0$ is continuous (hence integrable) on $[0,1]$. By part 1 of the Fundamental Theorem of Calculus, the function $\displaystyle\int_r^1 f(x)\,dx$ is a continuous function of r for $r\in[0,1]$ and so we have

$$\int_0^1 \frac{\sin x}{x}\,dx = \lim_{r\to 0^+}\int_r^1 \frac{\sin x}{x}\,dx = \lim_{r\to 0^+}\int_r^1 f(x)\,dx = \int_0^1 f(x)\,dx\,,$$

which is finite, so $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ converges.

Integrate by parts using $u=\frac{1}{x}$, $du = -\frac{1}{x^2}\,dx$, $v=-\sin x$ and $dv = \cos x\,dx$ to get

$$\int_1^\infty \frac{\sin x}{x}\,dx = \left[-\frac{\cos x}{x}\right]_1^\infty - \int_1^\infty \frac{\cos x}{x^2}\,dx = \cos(1) - \int_1^\infty \frac{\cos x}{x^2}\,dx\,.$$

Since $\left|\dfrac{\cos x}{x^2}\right| \leq \dfrac{1}{x^2}$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges, we see that $\displaystyle\int_1^\infty \left|\frac{\cos x}{x^2}\right| dx$ converges too, by comparison. Thus $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\,dx$ also converges by the Estimation Theorem.

Chapter 7. Series

Series

7.1 Definition: Let $\{a_n\}_{n\geq k}$ be a sequence. The **series** $\sum\limits_{n\geq k} a_n$ is defined to be the sequence $\{S_l\}_{l\geq k}$ where

$$S_l = \sum_{n=k}^{l} a_n = a_k + a_{k+1} + \cdots + a_l\,.$$

The term S_l is called the l^{th} **partial sum** of the series $\sum\limits_{n\geq k} a_n$. The **sum** of the series, denoted by

$$S = \sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + a_{k+2} + \cdots\,,$$

is the limit of the sequence of partial sums, if it exists, and we say the series **converges** when the sum exists and is finite.

7.2 Example: (Geometric Series) Show that for $a \neq 0$, the series $\sum\limits_{n\geq k} a_n$ converges if and only if $|r| < 1$, and that in this case

$$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}\,.$$

Solution: The l^{th} partial sum is

$$S_l = \sum_{n=k}^{\infty} ar^n = ar^k + ar^{k+1} + ar^{k+2} + \cdots + ar^l\,.$$

When $r = 1$ we have $S_l = a(l-k+1)$ and so $\lim\limits_{l\to\infty} S_l = \pm\infty$ ($+\infty$ when $a > 0$ and $-\infty$ when $a < 0$). When $r \neq 1$ we have $rS_l = ar^{k+1} + ar^{k+2} + \cdots + ar^l + ar^{l+1}$, so $S_l - rS_l = ar^k - ar^{l+1} = ar^k\big(1 - r^{l-k+1}\big)$ and so

$$S_l = \frac{ar^k(1-r^{l-k+1})}{1-r}\,.$$

When $r > 1$, $\lim\limits_{l\to\infty} r^{l-k+1} = \infty$ and so $\lim\limits_{l\to\infty} S_l = \pm\infty$ ($+\infty$ when $a > 0$ and $-\infty$ when $a < 0$). When $r \leq -1$, $\lim\limits_{l\to\infty} r^{l-k+1}$ does not exist, and so neither does $\lim\limits_{l\to\infty} S_l$. When $|r| < 1$, we have $\lim\limits_{l\to\infty} r^{l-k+1} = 0$ and so $\lim\limits_{l\to\infty} S_l = \dfrac{ar^k}{1-r}$, as required.

7.3 Example: Find $\displaystyle\sum_{n=-1}^{\infty} \frac{3^{n+1}}{2^{2n-1}}$.

Solution: This is a geometric series. By the formula in the previous example, we have

$$\sum_{n=-1}^{\infty} \frac{3^{n+1}}{2^{2n-1}} = \sum_{n=-1}^{\infty} \frac{3\cdot 3^n}{2^{-1}\cdot 4^n} = \sum_{n=-1}^{\infty} 6\left(\tfrac{3}{4}\right)^n = \frac{6\left(\frac{3}{4}\right)^{-1}}{1-\frac{3}{4}} = \frac{6\cdot\frac{4}{3}}{\frac{1}{4}} = 32\,.$$

7.4 Example: (Telescoping Series) Find $\sum_{i=1}^{\infty} \frac{1}{n^2+2n}$.

Solution: We use a partial fractions decomposition. The l^{th} partial sum is

$$\begin{aligned}
S_l &= \sum_{n=1}^{l} \frac{1}{n(n+2)} = \sum_{n=1}^{l} \left(\frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2} \right) = \tfrac{1}{2} \sum_{n=1}^{l} \left(\tfrac{1}{n} - \tfrac{1}{n+2} \right) \\
&= \tfrac{1}{2}\Big(\left(1 - \tfrac{1}{3}\right) + \left(\tfrac{1}{2} - \tfrac{1}{4}\right) + \left(\tfrac{1}{3} - \tfrac{1}{5}\right) + \cdots + \left(\tfrac{1}{n-2} - \tfrac{1}{n}\right) + \left(\tfrac{1}{n-1} - \tfrac{1}{n+1}\right) + \left(\tfrac{1}{n} - \tfrac{1}{n+2}\right) \Big) \\
&= \tfrac{1}{2}\left(1 + \tfrac{1}{2} - \tfrac{1}{n+1} - \tfrac{1}{n+2}\right) ,
\end{aligned}$$

since all the other terms cancel. Thus the sum of the series is

$$S = \lim_{l \to \infty} S_l = \tfrac{1}{2}\left(1 + \tfrac{1}{2}\right) = \tfrac{3}{4}\,.$$

7.5 Theorem: *(First Finitely Many Terms do Not Affect Convergence) Let $\{a_n\}_{n \geq k}$ be a sequence. Then for any integer $m \geq k$, the series $\sum_{n \geq k} a_n$ converges if and only if the series $\sum_{n \geq m} a_n$ converges, and in this case*

$$\sum_{n=k}^{\infty} a_n = \big(a_k + a_{k+1} + \cdots + a_{m-1}\big) + \sum_{n=m}^{\infty} a_n\,.$$

Proof: Let $S_l = \sum_{n=k}^{l} a_n$ and let $T_l = \sum_{n=m}^{l} a_n$. Then for all $l \geq m$ we have

$$S_l = \big(a_k + a_{k+1} + \cdots + a_{m-1}\big) + T_l\,,$$

and so $\{S_l\}$ converges if and only if $\{T_l\}$ converges, and in this case

$$\lim_{l \to \infty} S_l = \big(a_k + a_{k+1} + \cdots + a_{m-1}\big) + \lim_{l \to \infty} T_l\,.$$

7.6 Note: Since the first finitely many terms do not affect the convergence of a series, we often omit the subscript $n \geq k$ in the expression $\sum_{n \geq k} a_n$ when we are interested in whether or not the series converges. On the other hand, we cannot omit the subscript $n = k$ when we are interested in the value of the sum $\sum_{n=k}^{\infty} a_n$.

7.7 Definition: When we approximate a value x by the value y, the (absolute) **error** in our approximation is $|x - y|$.

7.8 Note: If $\sum_{n \geq k} a_n$ converges and $l \geq k$ then, by the above theorem, so does $\sum_{n \geq l+1}^{\infty} a_n$. If we approximate the sum $S = \sum_{n=k}^{\infty} a_n$ by the l^{th}partial sum $S_l = \sum_{n=k}^{l} a_n$, then the **error** in our approximation is

$$\big|S - S_l\big| = \left| \sum_{n=l+1}^{\infty} a_n \right|\,.$$

7.9 Theorem: *(Linearity) If* $\sum a_n$ *and* $\sum b_n$ *are convergent series then*

(1) for any real number c, $\sum ca_n$ *converges and* $\displaystyle\sum_{n=k}^{\infty} ca_n = c\sum_{n=k}^{\infty} a_n$ *, and*

(2) the series $\sum(a_n + b_n)$ *converges and* $\displaystyle\sum_{n=k}^{\infty}(a_n + b_n) = \sum_{n=k}^{\infty} a_n + \sum_{n=k}^{\infty} b_n$.

Proof: This follows immediately from the Linearity Theorem for sequences.

7.10 Theorem: *(Series of Positive Terms) Let* $\sum a_n$ *be a series.*

(1) If $a_n \geq 0$ *for all* $n \geq k$ *then either* $\sum a_n$ *converges or* $\displaystyle\sum_{n=k}^{\infty} a_n = \infty$.

(2) If $a_n \leq 0$ *for all* $n \geq k$ *then either* $\sum a_n$ *converges or* $\displaystyle\sum_{n=k}^{\infty} a_n = -\infty$.

Proof: This follows from the Monotone Convergence Theorem for sequences. Indeed if $a_n \geq 0$ for all $n \geq k$, then $\{S_l\}$ is increasing (since $S_{l+1} = S_l + a_{l+1} \geq S_l$ for all l). Either $\{S_l\}$ is bounded above, in which case $\{S_l\}$ converges hence $\sum a_n$ converges, or $\{S_l\}$ is unbounded, in which case $\displaystyle\lim_{n\to\infty} S_l = \infty$ hence $\displaystyle\sum_{n=k}^{\infty} a_n = \infty$.

7.11 Theorem: *(Divergence Test) If* $\sum a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$*. Equivalently, if* $\lim_{n\to\infty} a_n$ *either does not exist, or exists but is not equal to 0, then* $\sum a_n$ *diverges.*

Proof: Suppose that $\sum a_n$ converges, and say $\sum_{n=k}^{\infty} a_n = S$. Let S_l be the l^{th}partial sum. Then $\lim_{l\to\infty} S_l = S = \lim_{l\to\infty} S_{l-1}$, and we have $a_l = S_l - S_{l-1}$, and so

$$\lim_{l\to\infty} a_l = \lim_{l\to\infty} S_l - \lim_{l\to\infty} S_{l-1} = S - S = 0\,.$$

7.12 Example: Determine whether $\sum e^{1/n}$ converges.

Solution: Since $\lim_{n\to\infty} e^{1/n} = e^0 = 1$, $\sum e^{1/n}$ diverges by the Divergence Test.

7.13 Note: The converse of the Divergence Test is false. For example, as we shall see in Example 6.27 below, $\sum \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

7.14 Theorem: *(Integral Test) Let* $f(x)$ *be positive and decreasing for* $x \geq k$*, and let* $a_n = f(n)$ *for all integers* $n \geq k$*. Then* $\sum a_n$ *converges if and only if* $\int_k^\infty f(x)\,dx$ *converges, and in this case, for any* $l \geq k$ *we have*

$$\int_{l+1}^\infty f(x)\,dx \leq \sum_{n=l+1}^\infty a_n \leq \int_l^\infty f(x)\,dx\,.$$

Proof: Let T_m be the m^{th} partial sum for $\sum_{n\geq l+1} a_n$, so $T_m = \sum_{n=l+1}^m a_n$. Note that since $f(x)$ is decreasing, it is integrable on any closed interval. Also, for each $n \geq l$ we have $a_n = f(n) \leq f(x)$ for all $x \in [n-1, n]$, so $\int_{n-1}^n f(x)\,dx \geq \int_{n-1}^n a_n\,dx = a_n$ and so

$$T_m = \sum_{n=l+1}^m a_n \leq \sum_{n=l+1}^m \int_{n-1}^n f(x)\,dx = \int_l^m f(x)\,dx \leq \int_l^\infty f(x)\,dx\,.$$

Since $f(n) = a_n$ is positive, the sequence $\{T_m\}$ is increasing. If $\int_k^\infty f$ converges, then $\{T_n\}$ is bounded above by $\int_l^\infty f(x)\,dx$, and so it converges with $\lim_{m\to\infty} T_m \leq \int_l^\infty f(x)\,dx$.

Similarly, for each $n \geq l$ we have $a_n = f(n) \geq f(x)$ for all $x \in [n, n+1]$ so that $\int_n^{n+1} f(x)\,dx \leq \int_n^{n+1} a_n dx = a_n$ and so

$$T_m = \sum_{n=l+1}^m a_n \geq \sum_{n=l+1}^m \int_n^{n+1} f(x)\,dx = \int_{l+1}^{m+1} f(x)\,dx\,.$$

If $\int_k^\infty f$ converges, then $\lim_{m\to\infty} T_m \geq \lim_{m\to\infty} \int_{l+1}^{m+1} f(x)\,dx = \int_{l+1}^\infty f(x)\,dx$. If $\int_k^\infty f = \infty$ then $\lim_{m\to\infty} \int_{l+1}^{m+1} f(x)\,dx = \infty$, and so $\lim_{m\to\infty} T_m = \infty$ too, by Comparison.

7.15 Example: (p-Series) Show that the series $\sum_{n\geq 1} \dfrac{1}{n^p}$ converges if and only if $p > 1$. In particular, the **harmonic series** $\sum \frac{1}{n}$ diverges.

Solution: If $p < 0$ then $\lim_{n\to\infty} \dfrac{1}{n^p} = \infty$ and if $p = 0$ then $\lim_{n\to\infty} \dfrac{1}{n^p} = 1$, so in either case $\sum \frac{1}{n^p}$ diverges by the Divergence Test. Suppose that $p > 0$. Let $a_n = \frac{1}{n^p}$ for integers $n \geq 1$, and let $f(x) = \frac{1}{x^p}$ for real numbers $x \geq 1$. Note that $f(x)$ is positive and decreasing for $x \geq 1$ and $a_n = f(n)$ for all $n \geq 1$. Since we know that $\displaystyle\int_1^\infty f(x)\,dx$ converges if and only if $p > 1$, it follows from the Integral Test that $\sum a_n$ converges if and only if $p > 1$.

7.16 Example: Approximate $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2}$ so that the error is at most $\frac{1}{100}$.

Solution: We let $a_n = \frac{1}{2n^2}$ and $f(x) = \frac{1}{2x^2}$ so that we can apply the Integral Test. If we choose to approximate the sum S by the l^{th}partial sum S_l, then the error is

$$E = S - S_l = \sum_{n=l+1}^{\infty} a_n \leq \int_l^\infty \frac{1}{2x^2}\,dx = \left[-\frac{1}{2x}\right]_l^\infty = \frac{1}{2l}\,,$$

and so to insure that $E \leq \frac{1}{100}$ we can choose l so that $\frac{1}{2l} \leq \frac{1}{100}$, that is $l \geq 50$. Since it would be tedious to add up the first 50 terms of the series, we take an alternate approach. The Integral Test gives us upper and lower bounds: we have

$$\begin{aligned} \int_{l+1}^\infty f(x)\,dx \leq S - S_l &\leq \int_l^\infty f(x)\,dx \\ \frac{1}{2(l+1)} \leq S - S_l &\leq \frac{1}{2l} \\ S_l + \frac{1}{2(l+1)} \ \leq S &\leq \ S_l + \frac{1}{2l}\,. \end{aligned}$$

If approximate S using the midpoint of the upper and lower bounds, that is if we make the approximation $S \cong S_l + \frac{1}{2}\left(\frac{1}{2l} + \frac{1}{2(l+1)}\right)$, then the error E will be at most half of the difference of the bounds:

$$E \leq \tfrac{1}{2}\left(\tfrac{1}{2l} - \tfrac{1}{2(l+1)}\right) = \tfrac{1}{4l(l+1)}\,.$$

To get $E \leq \frac{1}{100}$ we want $\frac{1}{4\,l\,(l+1)} \leq \frac{1}{100}$, that is $l\,(l+1) \geq 25$, and so we can take $l = 5$. Thus we estimate

$$S \cong S_5 + \tfrac{1}{2}\left(\tfrac{1}{10} + \tfrac{1}{12}\right) = \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{18} + \tfrac{1}{32} + \tfrac{1}{50} + \tfrac{1}{20} + \tfrac{1}{24} = \tfrac{5929}{7200}\,.$$

$\big($Incidentally, the exact value of this sum is $\frac{\pi^2}{12}\big)$.

7.17 Theorem: *(Comparison Test) Let $0 \le a_n \le b_n$ for all $n \ge k$. Then if $\sum b_n$ converges then so does $\sum a_n$ and in this case,*

$$\sum_{n=k}^{\infty} a_n \le \sum_{n=k}^{\infty} b_n \,.$$

Proof: Let $S_l = \sum_{n=k}^{l} a_n$ and let $T_l = \sum_{n=k}^{l} b_n$. Since $0 \le a_n, b_n$ for all n, the sequences $\{S_l\}$ and $\{T_l\}$ are increasing. Since $a_n \le b_n$ for all n we have $S_l \le T_l$ for all l. Suppose that $\sum b_n$ converges with say $\sum_{n=k}^{\infty} b_n = T$ so that $\lim_{l\to\infty} \{T_l\} = T$. Then $S_l \le T_l \le T$ for all l, so $\{S_l\}$ is increasing and bounded above, hence convergent, and $\lim_{l\to\infty} S_l \le \lim_{l\to\infty} T_l$.

7.18 Example: Determine whether $\sum_{n\ge 0} \dfrac{1}{\sqrt{n^3+1}}$ converges.

Solution: Note that $0 \le \frac{1}{\sqrt{n^3+1}} \le \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$ for all $n \ge 1$, and $\sum \frac{1}{n^{3/2}}$ converges since it is a p-series with $p = \frac{3}{2}$, and so $\sum \frac{1}{\sqrt{n^3+1}}$ also converges, by the Comparison Test.

7.19 Example: Determine whether $\sum_{n\ge 1} \tan \frac{1}{n}$ converges.

Solution: For $0 < x < \frac{\pi}{2}$ we have $x < \tan x$, so for $n \ge 1$ we have $0 < \frac{1}{n} < \tan \frac{1}{n}$. Since the harmonic series $\sum \frac{1}{n}$ diverges, the series $\sum \tan \frac{1}{n}$ also diverges by the Comparison Test.

7.20 Example: Approximate $S = \sum_{n=0}^{\infty} \dfrac{1}{n!}$ so that the error is at most $\frac{1}{100}$.

Solution: If we make the approximation $S \cong S_l = \sum_{n=0}^{l} \dfrac{1}{n!}$ then the error is

$$\begin{aligned}
E = S - S_l &= \sum_{n=l+1}^{\infty} \frac{1}{n!} \\
&= \tfrac{1}{(l+1)!} + \tfrac{1}{(l+2)!} + \tfrac{1}{(l+3)!} + \tfrac{1}{(l+4)!} + \cdots \\
&= \tfrac{1}{(l+1)!}\left(1 + \tfrac{1}{l+2} + \tfrac{1}{(l+2)(l+3)} + \tfrac{1}{(l+2)(l+3)(l+4)} + \cdots\right) \\
&\le \tfrac{1}{(l+1)!}\left(1 + \tfrac{1}{l+2} + \tfrac{1}{(l+2)^2} + \tfrac{1}{(l+2)^3} + \cdots\right) \\
&= \tfrac{1}{(l+1)!}\,\frac{1}{1 - \frac{1}{l+2}} \\
&= \tfrac{l+2}{(l+1)(l+1)!}
\end{aligned}$$

where we used the Comparison Test and the formula for the sum of a geometric series. To get $E \le \frac{1}{100}$ we can choose l so that $\frac{l+2}{(l+1)(l+1)!} \le \frac{1}{100}$. By trial and error, we find that we can take $l = 4$, so we make the approximation

$$S \cong S_4 = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} = \tfrac{65}{24}\,.$$

(Incidentally, we shall see later that the exact value of this sum is e).

7.21 Theorem: *(Limit Comparison Test) Let $a_n \geq 0$ and let $b_n > 0$ for all $n \geq k$. Suppose that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = r$. Then*
(1) if $r = \infty$ and $\sum a_n$ converges then so does $\sum b_n$,
(2) if $r = 0$ and $\sum b_n$ converges then so does $\sum a_n$, and
(3) if $0 < r < \infty$ then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof: If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$, then for large n we have $\frac{a_n}{b_n} > 1$ so that $a_n > b_n$, and so if $\sum a_n$ converges, then so does $\sum b_n$ by the Comparison Test. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ then for large n we have $\frac{a_n}{b_n} < 1$ so $a_n < b_n$, and so if $\sum b_n$ converges then so does $\sum a_n$ by the Comparison Test. Suppose that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = r$ with $0 < r < \infty$. Choose N so that when $n > N$ we have $\left|\frac{a_n}{b_n} - r\right| < \frac{r}{2}$ so that $\frac{r}{2} < \frac{a_n}{b_n} < \frac{3r}{2}$ and hence

$$0 < \tfrac{r}{2} b_n \leq a_n \leq \tfrac{3r}{2} b_n \,.$$

If $\sum a_n$ converges, then $\sum \frac{r}{2} b_n$ converges by the Comparison Test, and hence $\sum b_n$ converges by linearity. If $\sum b_n$ converges, then $\sum \frac{3r}{2} b_n$ converges by linearity, and hence so does $\sum a_n$ by the Comparison Test.

7.22 Example: Determine whether $\sum \frac{1}{\sqrt{n^3-1}}$ converges.

Solution: Note that we cannot use the same argument that we used earlier to show that $\sum \frac{1}{\sqrt{n^3+1}}$ converges, because $\frac{1}{\sqrt{n^3+1}} < \frac{1}{n^{3/2}}$ but $\frac{1}{\sqrt{n^3-1}} > \frac{1}{n^{3/2}}$. We use a different approach. Let $a_n = \frac{1}{\sqrt{n^3-1}}$ and let $b_n = \frac{1}{n^{3/2}}$. Then $\lim \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^{3/2}}{\sqrt{n^3-1}} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1 - \frac{1}{n^3}}} = 1$, and $\sum b_n = \sum \frac{1}{n^{3/2}}$ converges (its a p-series with $p = \frac{3}{2}$), and so $\sum a_n$ converges too, by the Limit Comparison Test.

7.23 Theorem: *(Ratio Test) Let $a_n > 0$ for all $n \geq k$. Suppose $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r$. Then*
(1) if $r < 1$ then $\sum a_n$ converges, and
(2) if $r > 1$ then $\lim_{n\to\infty} a_n = \infty$ so $\sum a_n = \infty$.

Proof: Suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$. Choose s with $r < s < 1$, and then choose N so that when $n > N$ we have $\frac{a_{n+1}}{a_n} < s$ and hence $a_{n+1} < s\,a_n$. Fix $k > N$. Then $a_{k+1} < sa_k$, $a_{k+2} < sa_{k+1} < s^2 a_k$, $a_{k+3} < sa_{k+2} < s^3 a_k$, and so on, so we have $a_n < b_n = s^{n-k} a_k$ for all $n \geq k$. Since $\sum b_n$ is geometric with ratio $s < 1$, it converges, and hence so does $\sum a_n$ by the Comparison Test.

Now suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r > 1$. Choose s with $1 < s < r$, then choose N so that when $n > N$ we have $\frac{a_{n+1}}{a_n} > s$ and hence $a_{n+1} > sa_n$. Fix $k > N$. Then as above $a_n > b_n = s^{n-k} a_k$ for all $n \geq k$, and $\lim_{n\to\infty} b_n = \infty$, so $\lim_{n\to\infty} a_n = \infty$ too.

7.24 Example: Determine whether $\sum \frac{5^n}{n!}$ converges.

Solution: Let $a_n = \frac{5^n}{n!}$. Then $\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5}{n+1} \to 0$ as $n \to \infty$, and so $\sum a_n$ converges by the Ratio Test.

7.25 Note: If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then $\sum a_n$ could converge or diverge. For example, if $a_n = \frac{1}{n}$ then $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \to 1$ as $n \to \infty$ and $\sum a_n$ diverges, but if $b_n = \frac{1}{n^2}$ then $\frac{b_{n+1}}{b_n} = \frac{n^2}{(n+1)^2} \to 1$ as $n \to \infty$ and $\sum b_n$ converges.

7.26 Theorem: *(Root Test) Let $a_n \geq 0$ for all $n \geq k$. Suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = r$. Then*
(1) if $r < 1$ then $\sum a_n$ converges, and
(2) if $r > 1$ then $\lim_{n\to\infty} a_n = \infty$ so $\sum a_n = \infty$.

Proof: The proof is left as an exercise. It is similar to the proof of the Ratio Test.

7.27 Example: Determine whether $\sum \left(\frac{n}{n+1}\right)^{n^2}$ converges.

Solution: Let $a_n = \left(\frac{n}{n+1}\right)^{n^2}$. Then $\sqrt[n]{a_n} = \left(\frac{n}{n+1}\right)^n = e^{n \ln\left(\frac{n}{n+1}\right)}$, and by l'Hôpital's Rule we have $\lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) = \lim_{x\to\infty} \dfrac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \dfrac{\frac{1}{(x+1)^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \dfrac{-x^2}{(x+1)^2} = -1$, and so $\lim_{n\to\infty} \sqrt[n]{a_n} = e^{-1} < 1$. Thus $\sum a_n$ converges by the Root Test.

7.28 Definition: A sequence $\{a_n\}_{n\ge k}$ is said to be **alternating** when either we have $a_n = (-1)^n|a_n|$ for all $n \ge k$ or we have $a_n = (-1)^{n+1}|a_n|$ for all $n \ge k$.

7.29 Theorem: *(Aternating Series Test) Let $\{a_n\}_{n\ge k}$ be an alternating series. If $\{|a_n|\}$ is decreasing with $\lim\limits_{n\to\infty} |a_n| = 0$ then $\sum\limits_{n\ge k} a_n$ converges, and in this case*

$$\left|\sum_{n=k}^{\infty} a_n\right| \le |a_k|\,.$$

Proof: To simplify notation, we give the proof in the case that $k = 0$ and $a_n = (-1)^n|a_n|$. Suppose that $\big\{|a_n|\big\}$ is decreasing with $|a_n| \to 0$. Let $S_l = \sum\limits_{n=0}^{l} a_n$. We consider the sequences $\{S_{2l}\}$ and $\{S_{2l-1}\}$ of even and odd partial sums. Note that since $\big\{|a_n|\big\}$ is decreasing, we have

$$S_{2l} - S_{2l-1} = |a_{2l}| - |a_{2l-1}| \le 0$$

so $\big\{S_{2l}\big\}$ is decreasing, and we have

$$\begin{aligned} S_{2l} &= |a_0| - |a_1| + |a_2| - |a_3| + \cdots + |a_{2l-2}| - |a_{2l-1}| + |a_{2l}| \\ &= \big(|a_0| - |a_1|\big) + \big(|a_2| - |a_3|\big) + \cdots + \big(|a_{2l-2}| - |a_{2l-1}|\big) + |a_{2l}| \\ &\ge |a_0| - |a_1| \end{aligned}$$

and so $\{S_{2l}\}$ is bounded below by $|a_0| - |a_1|$. Thus $\big\{S_{2l}\big\}$ converges by the Monotone Convergence Theorem. Similarly, $\big\{S_{2l-1}\big\}$ is increasing and bounded above by $|a_0|$, so it also converges, and we have $\lim\limits_{l\to\infty} S_{2l-1} \le |a_0|$.

Finally we note that since $|a_n| \to 0$, taking the limit on both sides of the equality $|a_{2l}| = S_{2l} - S_{2l-1}$ gives $0 = \lim\limits_{l\to\infty} S_{2l} - \lim\limits_{l\to\infty} S_{2l-1}$. and so we have $\lim\limits_{l\to\infty} S_{2l} = \lim\limits_{l\to\infty} S_{2l-1}$. It follows that $\big\{S_l\big\}$ converges, and we have $\lim\limits_{l\to\infty} S_l = \lim\limits_{l\to\infty} S_{2l} = \lim\limits_{l\to\infty} S_{2l-1} \le |a_0|$.

7.30 Example: Determine whether $\sum\limits_{n\ge 2} \dfrac{(-1)^n \ln n}{\sqrt{n}}$ converges.

Solution: Let $a_n = \dfrac{(-1)^n \ln n}{\sqrt{n}}$. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$ so that $|a_n| = f(n)$. Note that

$$f'(x) = \frac{\frac{1}{x}\cdot\sqrt{x} - \ln x \cdot \frac{1}{2\sqrt{x}}}{x} = \frac{2 - \ln x}{2x^{3/2}}\,,$$

so we have $f'(x) < 0$ for $x > e^2$. Thus $f(x)$ is decreasing for $x > e^2$, and so $\big\{|a_n|\big\}$ is decreasing for $n \ge 8$. Also, by l'Hôpital's Rule, we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \tfrac{2}{\sqrt{x}} = 0$$

and so $|a_n| \to 0$ as $n \to \infty$. Thus $\sum a_n$ converges by the Alternating Series Test.

7.31 Example: Approximate the sum $S = \sum_{n=0}^{\infty} \frac{(-2)^n}{(2n)!}$ so that the error is at most $\frac{1}{2000}$.

Solution: Let $a_n = \frac{(-2)^n}{(2n)!}$. Note that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{2^n} = \frac{2}{(2n+2)(2n+1)} = \frac{1}{(n+1)(2n+1)}\,.$$

Since $\frac{|a_{n+1}|}{|a_n|} \leq 1$ for all $n \geq 0$, we know that $\{|a_n|\}$ is decreasing. Since $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$, we know that $\sum |a_n|$ converges by the Ratio Test, and so $|a_n| \to 0$ by the Divergence Test. This shows that we can apply the Alternating Series Test.

If we approximate S by the l^{th}partial sum $S_l = \sum_{n=0}^{l} a_n$, then by the Alternating Series Test, the error is

$$E = \big|S - S_l\big| = \left|\sum_{n=l+1}^{\infty} a_n\right| \leq \big|a_{l+1}\big| = \frac{2^{l+1}}{(2l+2)!}\,.$$

To get $E \leq \frac{1}{2000}$ we can choose l so that $\frac{2^{l+1}}{(l+1)!} \leq \frac{1}{2000}$. By trial and error we find that we can take $l = 3$. Thus we make the approximation

$$S \cong S_3 = 1 - \tfrac{2}{2!} + \tfrac{2^2}{4!} - \tfrac{2^3}{6!} = 1 - 1 + \tfrac{1}{6} + \tfrac{1}{90} = \tfrac{7}{45}\,.$$

$\big($We shall see later that the exact value of this sum is $\cos\sqrt{2}\big)$.

7.32 Definition: A series $\sum\limits_{n\geq k} a_n$ is said to **converge absolutely** when $\sum\limits_{n\geq k} |a_n|$ converges. The series is said to **converge conditionally** if $\sum\limits_{n\geq k} a_n$ converges but $\sum\limits_{n\geq k} |a_n|$ diverges.

7.33 Example: For $0 < p \leq 1$, the p-series $\sum \frac{1}{n^p}$ diverges, but since $\left\{\frac{1}{n^p}\right\}$ is decreasing towards 0, $\sum \frac{(-1)^n}{n^p}$ converges by the Alternating Series Test. Thus for $0 < p \leq 1$, the alternating p-series $\sum \frac{(-1)^n}{n^p}$ converges conditionally.

7.34 Theorem: *(Absolute Convergence Implies Convergence) If $\sum |a_n|$ converges then so does $\sum a_n$.*

Proof: Suppose that $\sum |a_n|$ converges. Note that $-|a_n| \leq a_n \leq |a_n|$ so that

$$0 \leq a_n + |a_n| \leq 2|a_n| \text{ for all } n\,.$$

Since $\sum |a_n|$ converges, $\sum 2|a_n|$ converges by linearity, and so $\sum \big(a_n + |a_n|\big)$ converges by the Comparison Test. Since $\sum |a_n|$ and $\sum \big(a_n + |a_n|\big)$ both converge, $\sum a_n$ converges by linearity.

7.35 Example: Determine whether $\sum \frac{\sin n}{n^2}$ converges.

Solution: Let $a_n = \frac{\sin n}{n^2}$. Then $|a_n| = \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges (its a p-series with $p = 2$), $\sum |a_n|$ converges by the Comparison Test, and hence $\sum a_n$ converges too, since absolute convergence implies convergence.

Multiplication of Series

7.36 Theorem: *(Multiplication of Series) Suppose that* $\sum\limits_{n\geq 0}|a_n|$ *converges and* $\sum\limits_{n\geq 0} b_n$ *converges and define* $c_n=\sum\limits_{k=0}^{n} a_k b_{n-k}$*. Then* $\sum\limits_{n\geq 0} c_n$ *converges and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Proof: Let $A_l=\sum\limits_{n=0}^{l} a_n$, $B_l=\sum\limits_{n=0}^{l} b_n$, $C_l=\sum\limits_{n=0}^{l} c_n$, $A=\sum\limits_{n=0}^{\infty} a_n$, $B=\sum\limits_{n=0}^{\infty} b_n$, $K=\sum\limits_{n=0}^{\infty}|a_n|$ and $E_l=B-B_l$. Then we have

$$\begin{aligned} C_l &= a_0b_0+(a_0b_1+a_1b_0)+(a_0b_2+a_1b_1+a_2b_0)+\cdots+(a_0b_l+\cdots+a_lb_0)\\ &= a_0B_l+a_1B_{l-1}+a_2B_{l-2}+\cdots+a_lB_0\\ &= a_0(B-E_l)+a_1(B-E_{l-1})+\cdots+a_l(B-E_0)\\ &= A_lB-\big(a_0E_l+a_1E_{l-1}+\cdots+a_lE_0\big) \end{aligned}$$

and so

$$\big|AB-C_l\big| \leq \big|(A-A_l)B\big| + \big|a_0E_l+a_1E_{l-1}+\cdots+a_lE_0\big|.$$

Let $\epsilon>0$. Choose m so that $j>m \Longrightarrow E_j<\frac{\epsilon}{3K}$. Let $E=\max\big\{|E_0|,\cdots,|E_m|\big\}$. Choose $L>m$ so that when $l>L$ we have $\sum\limits_{n=l-m}^{l}|a_n|<\frac{\epsilon}{3E}$ and we have $|A_l-A||B|<\frac{\epsilon}{3}$. Then for $l>L$,

$$\begin{aligned} \big|C_l-AB\big| &< \big|(A_l-A)B\big| + \big|a_0E_l+\cdots+a_{l-m-1}E_{m+1}\big| + \big|a_{l-m}E_m+\cdots+a_lE_0\big|\\ &\leq \tfrac{\epsilon}{3} + \left(\sum_{n=0}^{l-m-1}|a_n|\right)\tfrac{\epsilon}{3K} + \left(\sum_{n=l-m+1}^{l}|a_n|\right)E\\ &< \tfrac{\epsilon}{3} + K\tfrac{\epsilon}{3K} + \tfrac{\epsilon}{3E}E = \epsilon\,. \end{aligned}$$

7.37 Example: Find an example of sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ such that $\sum\limits_{n\geq 0} a_n$ and $\sum\limits_{n\geq 0} b_n$ both converge, but $\sum\limits_{n\geq 0} c_n$ diverges where $c_n=\sum\limits_{k=0}^{n} a_k b_{n-k}$.

Solution: Let $a_n=b_n=\dfrac{(-1)^n}{\sqrt{n+1}}$ for $n\geq 0$, and let

$$c_n=\sum_{k=0}^{n} a_k b_{n-k} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}\,.$$

Recall that for $p,q\geq 0$ we have $\sqrt{pq}\leq \frac{1}{2}(p+q)$ (indeed $(p+q)^2-4pq=p^2-2pq+q^2=(p-q)^2\geq 0$, so $(p+q)^2\geq 4pq$). In particular $\sqrt{(k+1)(n-k+1)}\leq\frac{1}{2}(n+2)$ and so $|c_n|\geq\sum\limits_{k=0}^{n}\frac{2}{n+2}=\frac{2(n+1)}{n+2}$. Thus $\lim\limits_{n\to\infty}|c_n|\neq 0$ so $\sum c_n$ diverges by the Divergence Test.

Chapter 8. Power Series

Power Series

8.1 Definition: A **power series centred at** a is a series of the form

$$\sum_{n\geq 0} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

for some real numbers c_n, where we use the convention that $(x-a)^0 = 1$.

8.2 Example: The geometric series $\sum\limits_{n\geq 0} x^n$ is a power series centred at 0. It converges when $|x| < 1$ and for all such x the sum of the series is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,.$$

8.3 Theorem: *(The Interval and Radius of Convergence) Let $\sum\limits_{n\geq 0} c_n(x-a)^n$ be a power series. Then the set of $x \in \mathbf{R}$ for which the power series converges is an interval I centred at a. Indeed there exists a (possibly infinite) number $R \in [0,\infty]$ such that*

(1) if $|x-a| < R$ then $\sum\limits_{n\geq 0} c_n(x-a)^n$ converges absolutely, and

(2) if $|x-a| > R$ then $\sum\limits_{n\geq 0} c_n(x-a)^n$ diverges.

Proof: We prove parts (1) and (2) together by showing that for all $r > 0$, if $\sum c_n r^n$ converges then $\sum c_n(x-a)^n$ converges absolutely for all $x \in R$ with $|x-a| < r$ (we can then take R to be the least upper bound of the set of all such r). Let $r > 0$. Suppose that $\sum c_n r^n$ converges. Let $x \in \mathbf{R}$ with $|x-a| < r$. Choose s with $|x-a| < s < r$. Since $\sum c_n r^n$ converges, we have $c_n r^n \to 0$ by the Divergence Test. Choose $N > 0$ so that $|c_n r^n| \leq 1$ for all $n \geq N$. Then for $n \geq N$ we have

$$\left|c_n(x-a)^n\right| = |c_n r^n| \cdot \frac{|x-a|^n}{r^n} \leq \frac{|x-a|^n}{r^n} \leq \frac{s^n}{r^n} = \left(\tfrac{s}{r}\right)^n ,$$

and the series $\sum \left(\frac{s}{r}\right)^n$ converges (its geometric with positive ratio $\frac{s}{r} < 1$), and so the series $\sum \left|c_n(x-a)^n\right|$ converges too, by the Comparison Test.

8.4 Definition: The number R in the above theorem is called the **radius of convergence** of the power series, and the interval I is called the **interval of convergence** of the power series.

8.5 Example: Find the interval of convergence of the power series $\displaystyle\sum_{n\geq 1}\frac{(3-2x)^n}{\sqrt{n}}$.

Solution: First note that this is in fact a power series, since $\displaystyle\frac{(3-2x)^n}{\sqrt{n}} = \frac{(-2)^n}{\sqrt{n}}\left(x-\tfrac{3}{2}\right)^n$, and so $\displaystyle\sum_{n\geq 1}\frac{(3-2x)^n}{\sqrt{n}} = \sum_{n\geq 0}c_n(x-a)^n$, where $c_0=0$, $c_n=\frac{(-2)^n}{\sqrt{n}}$ for $n\geq 1$ and $a=\frac{3}{2}$. Now, let $a_n=\displaystyle\frac{(3-2x)^n}{\sqrt{n}}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(3-2x)^{n+1}}{\sqrt{n+1}}\,\frac{\sqrt{n}}{(3-2x)^n}\right| = \sqrt{\tfrac{n}{n+1}}\,|3-2x| \longrightarrow |3-2x| \text{ as } n\to\infty.$$

By the Ratio Test, $\sum a_n$ converges when $|3-2x|<1$ and diverges when $|3-2x|>1$. Equivalently, it converges when $x\in(1,2)$ and diverges when $x\notin[1,2]$. When $x=1$ so $(3-2x)=1$, we have $\sum a_n=\sum\frac{1}{\sqrt{n}}$, which diverges (its a p-series), and when $x=2$ so $(3-2x)=-1$, we have $\sum a_n=\sum\frac{(-1)^n}{\sqrt{n}}$ which converges by the Alternating Series Test. Thus the interval of convergence is $I=(1,2\,]$.

8.6 Note: An argument similar to the one used in the above example, using the Ratio Test, can be used to show that if $\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$ exists (finite or infinite) then the radius of convergence of the power series $\sum c_n(x-a)^n$ is equal to

$$R=\frac{1}{\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|}=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|\,.$$

Indeed if we let $R=\displaystyle\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$ and write $a_n=c_n(x-a)^n$ then we have

$$\frac{|a_{n+1}|}{|a_n|}=\frac{\left|c_{n+1}(x-a)^{n+1}\right|}{\left|c_n(x-a)^n\right|}=\left|\frac{c_{n+1}}{c_n}\right|\,|x-a|\longrightarrow\frac{1}{R}\,|x-a|$$

and so by the Ratio Test, if $|x-a|<R$ then $\sum|a_n|$ converges while if $|x-a|>R$ then $|a_n|\to\infty$ so $\sum a_n$ diverges. Thus R must be equal to the radius of convergence.

9. Operations on Power Series

9.1 Theorem: *(Continuity of Power Series) Suppose that the power series* $\sum c_n(x-a)^n$ *converges in an interval* I*. Then the sum* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *is continuous in* I*.*

Proof: We omit the proof

9.2 Theorem: *(Addition and Subtraction of Power Series) Suppose that the power series* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in the interval* I*. Then* $\sum(a_n+b_n)(x-a)^n$ *and* $\sum(a_n-b_n)(x-a)^n$ *both converge in* I*, and for all* $x \in I$ *we have*

$$\left(\sum_{n=0}^{\infty} a_n(x-a)^n\right) \pm \left(\sum_{n=0}^{\infty} b_n(x-a)^n\right) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x-a)^n \,.$$

Proof: This follows from Linearity.

9.3 Theorem: *(Multiplication of Power Series) Suppose the power series* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in an open interval* I *with* $a \in I$*. Let* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$*. Then* $\sum c_n(x-a)^n$ *converges in* I *and for all* $x \in I$ *we have*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \left(\sum_{n=0}^{\infty} a_n(x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-a)^n\right) \,.$$

Proof: This follows from the Multiplication of Series Theorem, since the power series converge absolutely in I.

9.4 Theorem: *(Division of Power Series) Suppose that* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in an open interval* I *with* $a \in I$*, and that* $b_0 \neq 0$*. Define* c_n *by*

$$c_0 = \tfrac{a_0}{b_0} \text{ , and for } n > 0,\ c_n = \tfrac{a_n}{b_0} - \tfrac{b_n c_0}{b_0} - \tfrac{b_{n-1}c_1}{b_0} - \cdots - \tfrac{b_1 c_{n-1}}{b_0} \,.$$

Then there is an open interval J *with* $a \in J$ *such that* $\sum c_n(x-a)^n$ *converges in* J *and for all* $x \in J$*,*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \frac{\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n}{\displaystyle\sum_{n=0}^{\infty} b_n(x-a)^n} \,.$$

Proof: We omit the proof.

9.5 Theorem: *(Composition of Power Series) Let* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *in an open interval* I *with* $a \in I$*, and let* $g(y) = \sum_{m=0}^{\infty} b_m(y-b)^m$ *in an open interval* J *with* $b \in J$ *and with* $a_0 \in J$*. Let* K *be an open interval with* $a \in K$ *such that* $f(K) \subset J$*. For each* $m \geq 0$*, let* $c_{n,m}$ *be the coefficients, found by multiplying power series, such that* $\sum_{n=0}^{\infty} c_{n,m}(x-a)^n = b_n\left(\sum_{n=0}^{\infty} a_n(x-a)^n - b\right)^m$*. Then* $\sum_{m\geq 0} c_{n,m}$ *converges for all* $m \geq 0$*, and for all* $x \in K$*,* $\sum_{n\geq 0}\left(\sum_{m=0}^{\infty} c_{n,m}\right)(x-a)^n$ *converges and*

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} c_{n,m}\right)(x-a)^n = g\big(f(x)\big)\,.$$

Proof: We omit the proof.

9.6 Theorem: *(Integration of Power Series) Supoose that* $\sum c_n(x-a)^n$ *converges in the interval* I*. Then for all* $x \in I$*, the sum* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *is integrable on* $[a,x]$ *(or* $[x,a]$*) and*

$$\int_a^x \sum_{n=0}^{\infty} c_n(t-a)^n\,dt = \sum_{n=0}^{\infty}\int_a^x c_n(t-a)^n\,dt = \sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1}\,.$$

Proof: We omit the proof

9.7 Theorem: *(Differentiation of Power Series) Suppose that* $\sum c_n(x-a)^n$ *converges in the open interval* I*. Then the sum* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *is differentiable in* I *and*

$$f'(x) = \sum_{n=1}^{\infty} n\,c_n(x-a)^{n-1}\,.$$

Proof: We omit the proof

9.8 Example: Find a power series centred at 0 whose sum is $f(x) = \dfrac{1}{x^2+3x+2}$, and find its interval of convergence.

Solution: We have

$$\begin{aligned} f(x) &= \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{1+x} - \frac{\frac{1}{2}}{1+\frac{x}{2}} \\ &= \sum_{n=0}^{\infty}(-x)^n - \sum_{n=0}^{\infty} \tfrac{1}{2}\left(-\tfrac{x}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n x^n - \sum_{n=0}^{\infty} \tfrac{(-1)^n}{2^{n+1}} x^n \\ &= \sum_{n=0}^{\infty}(-1)^n \left(1 - \tfrac{1}{2^{n+1}}\right) x^n \,. \end{aligned}$$

Since $\sum\limits_{n=0}^{\infty}(-x)^n$ converges if and only if $|x|<1$ and $\sum\limits_{n=0}^{\infty} \frac{1}{2}\left(-\frac{x}{2}\right)^n$ converges when $|x|<2$, it follows from Linearity the the sum of these two series converges if and only if $|x|<1$.

9.9 Example: Find a power series centred at -4 whose sum is $f(x) = \dfrac{1}{x^2+3x+2}$, and find its interval of convergence.

Solution: We have

$$\begin{aligned} f(x) &= \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{(x+4)-3} - \frac{1}{(x+4)-2} \\ &= \frac{-\frac{1}{3}}{1-\frac{x+4}{3}} + \frac{\frac{1}{2}}{1-\frac{x+4}{2}} = \sum_{n=0}^{\infty} -\tfrac{1}{3}\left(\tfrac{x+4}{3}\right)^n + \sum_{n=0}^{\infty} \tfrac{1}{2}\left(\tfrac{x+4}{2}\right)^n \\ &= \sum_{n=0}^{\infty}\left(\tfrac{1}{2^{n+1}} - \tfrac{1}{3^{n+1}}\right)(x+4)^n \,. \end{aligned}$$

Since $\sum\limits_{n=0}^{\infty} -\frac{1}{3}\left(\frac{x+4}{3}\right)^n$ converges when $|x+4|<3$ and $\sum\limits_{n=0}^{\infty} \frac{1}{2}\left(\frac{x+4}{2}\right)^n$ converges if and only if $|x+4|<2$, it follows that their sum converges if and only if $|x+4|<2$.

9.10 Example: Find a power series centred at 0 whose sum is $f(x) = \dfrac{1}{(1-x)^2}$.

Solution: We provide three solutions. For the first solution, we multiply two power series. For $|x|<1$ we have

$$\begin{aligned} f(x) &= \frac{1}{1-x} \cdot \frac{1}{1-x} \\ &= \left(1+x+x^2+x^3+\cdots\right)\left(1+x+x^2+x^3+\cdots\right) \\ &= 1 + (1+1)x + (1+1+1)x^2 + (1+1+1+1)x^3 + \cdots \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &= \sum_{n=0}^{\infty}(n+1)x^n \,. \end{aligned}$$

For the second solution, we note that $f(x) = \dfrac{1}{1-2x+x^2}$ and we use long division.

$$\begin{array}{r l}
 & 1+2x+3x^2+4x^3+5x^4+\cdots \\ \cline{2-2}
1-2x+x^2 \;\big) & 1+0x+0x^2+0x^3+0x^4-\cdots \\
 & \underline{1-2x+x^2} \\
 & \quad\;\; 2x-x^2 \\
 & \quad\;\; \underline{2x-4x^2+2x^3} \\
 & \qquad\quad 3x^2-2x^3 \\
 & \qquad\quad \underline{3x^2-6x^3+3x^4} \\
 & \qquad\qquad\quad 4x^3-8x^4+\cdots \\
 & \qquad\qquad\quad \underline{4x^3-8x^4+\cdots} \\
 & \qquad\qquad\qquad\quad 5x^4+\cdots
\end{array}$$

For the third solution, we note that $\displaystyle\int \frac{1}{(1-x)^2} = \frac{1}{1-x}$ and we use differentiation.

$$\begin{aligned}
\frac{1}{1-x} &= 1+x^2+x^3+x^4+x^5+\cdots \\
\frac{d}{dx}\left(\frac{1}{1-x}\right) &= \frac{d}{dx}\big(1+x+x^2+x^3+x^4+x^5+\cdots\big) \\
\frac{1}{(1-x)^2} &= 1+2x+3x^2+4x^3+5x^4+\cdots\,.
\end{aligned}$$

9.11 Example: Find a power series centred at 0 whose sum is $\ln(1+x)$.

Solution: For $|x|<1$ we have

$$\begin{aligned}
\frac{1}{1+x} &= 1-x+x^2-x^3+\cdots \\
\ln(1+x) &= \int 1-x+x^2-x^3++\cdots\;dx \\
&= c+x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\cdots
\end{aligned}$$

Putting in $x=0$ gives $0=c$, and so

$$\ln(1+x) = \sum_{n=1}^{\infty} \tfrac{(-1)^{n+1}}{n}x^n = x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\cdots\,.$$

9.12 Example: Find a power series centred at 0 whose sum is $f(x)=\tan^{-1}x$.

Solution: For $|x|<1$ we have

$$\begin{aligned}
\frac{1}{1+x^2} &= 1-x^2+x^4-x^6+\cdots \\
\tan^{-1}x &= \int 1-x^2+x^4-x^6+\cdots\;dx \\
&= c+x-\tfrac{1}{3}\,x^3+\tfrac{1}{5}\,x^5-\tfrac{1}{7}\,x^7+\cdots
\end{aligned}$$

Putting in $x=0$ gives $0=c$, and so

$$\tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}\,x^{2n+1} = x-\tfrac{1}{3}\,x^3+\tfrac{1}{5}\,x^5-\tfrac{1}{7}\,x^7+\cdots$$

Chapter 9. Taylor Series

Taylor Series

9.1 Theorem: *Suppose that* $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ *in an open interval* I *centred at* a. *Then* f *is infinitely differentiable at* a *and for all* $n \geq 0$ *we have*

$$c_n = \frac{f^{(n)}(a)}{n!},$$

where $f^{(n)}(a)$ *denotes the* n^{th} *derivative of* f *at* a.

Proof: By repeated application of the Differentiation of Power Series Theorem, for all $x \in I$, we have

$$f'(x) = \sum_{n=1}^{\infty} n\, c_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)\, c_n (x-a)^{n-2}$$

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)\, c_n (x-a)^{n-3},$$

and in general

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, c_n (x-a)^{n-k}$$

and so $f(a) = c_0$, $f'(a) = c_1$, $f''(a) = 2 \cdot 1\, c_2$ and $f'''(a) = 3 \cdot 2 \cdot 1\, c_3$, and in general

$$f^{(n)}(a) = n!\; c_n$$

9.2 Definition: Given a function $f(x)$ whose derivatives of all order exist at $x = a$, we define the **Taylor series** of $f(x)$ centred at a to be the power series

$$T(x) = \sum_{n \geq 0} c_n (x-a)^n \quad \text{where } c_n = \frac{f^{(n)}(a)}{n!}$$

and we define the l^{th} **Taylor Polynomial** of $f(x)$ centred at a to be the l^{th} partial sum

$$T_l(x) = \sum_{n=0}^{l} c_n (x-a)^n \quad \text{where } c_n = \frac{f^{(n)}(a)}{n!}$$

9.3 Example: Find the Taylor series centred at 0 for $f(x) = e^x$.

Solution: We have $f^{(n)}(x) = e^x$ for all n, so $f^{(n)}(0) = 1$ and $c_n = \frac{1}{n!}$ for all $n \geq 0$. Thus the Taylor series is

$$T(x) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\, x^n = 1 + x + \tfrac{1}{2!}x^2 = \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \cdots.$$

9.4 Example: Find the Taylor series centred at 0 for $f(x) = \sin x$.

Solution: We have $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f''''(x) = \sin x$ and so on, so that in general $f^{(2n)}(x) = (-1)^n \sin x$ and $f^{(2n+1)}(x) = (-1)^n \cos x$. It follows that $f^{(2n)}(0) = 0$ and $f^{(2n+1)}(0) = (-1)^n$, so we have $c_{2n} = 0$ and $c_{2n+1} = \frac{(-1)^n}{(2n+1)!}$. Thus

$$T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots .$$

9.5 Example: Find the Taylor series centred at 0 for $f(x) = \cos x$.

Solution: We have $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f''''(x) = \cos x$ and so on, so that in general $f^{(2n)}(x) = (-1)^n \cos x$ and $f^{(2n+1)}(x) = (-1)^{n+1} \sin x$. It follows that $f^{(2n)}(0) = (-1)^n$ and $f^{(2n+1)}(0) = 0$, so we have $c_{2n} = \frac{(-1)^n}{(2n)!}$ and $c_{2n+1} = 0$. Thus

$$T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{5!}x^6 + \cdots .$$

9.6 Example: Find the Taylor series centred at 0 for $f(x) = (1+x)^p$ where $p \in \mathbf{R}$.

Solution: $f'(x) = p(1+x)^{p-1}$, $f''(x) = p(p-1)(1+x)^{p-2}$, $f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$, and in general

$$f^{(n)}(x) = p(p-1)(p-2)\cdots(p-n+1)(1+x)^{p-n}\,,$$

so $f(0) = 1$, $f'(0) = p$, $f''(0) = p(p-1)$, and in general $f^{(n)}(0) = p(p-1)(p-2)\cdots(p-n+1)$, and so we have $c_n = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$. Thus the Taylor series is

$$T(x) = \sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \frac{p(p-1)(p-2)(p-3)}{4!}x^4 + \cdots$$

where we use the notation

$$\binom{p}{0} = 1\ \text{, and for } n \ge 1,\ \binom{p}{n} = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$$

9.7 Theorem: *(Taylor) Let $f(x)$ be infinitely differentiable in an open interval I with $a \in I$. Let $T_l(x)$ be the l^{th} Taylor polynomial for $f(x)$ centred at a. Then for all $x \in I$ there exists a number c between a and x such that*

$$f(x) - T_l(x) = \frac{f^{(l+1)}(c)}{(l+1)!}(x-a)^{l+1}\,.$$

Proof: When $x = a$ both sides of the above equation are 0. Suppose that $x > a$ (the case that $x < a$ is similar). Since $f^{(l+1)}$ is differentiable and hence continuous, by the Extreme Value Theorem it attains its maximum and minimum values, say M and m. Since $m \le f^{(l+1)}(t) \le M$ for all $t \in I$, we have

$$\int_a^{t_1} m\,dt \le \int_a^{t_1} f^{(l+1)}(t)\,dt \le \int_a^{t_1} M\,dt$$

that is

$$m(t_1 - a) \le f^{(l)}(t_1) - f^{(l)}(a) \le M(t_1 - a)$$

for all $t_1 > a$ in I. Integrating each term with respect to t_1 from a to t_2, we get

$$\tfrac{1}{2}m(t_2-a)^2 \le f^{(l-1)}(t_2) - f^{(l)}(a)(t_2-a) \le \tfrac{1}{2}M(t_t-a)^2$$

for all $t_2 > a$ in I. Integrating with respect to t_2 from a to t_3 gives

$$\tfrac{1}{3!}m(t_3-a)^3 \le f^{(l-2)}(t_3) - f^{(l-2)}(a) - \tfrac{1}{2}f^{(l)}(a)(t_3-a)^3 \le \tfrac{1}{3!}M(t_3-a)^3$$

for all $t_3 > a$ in I. Repeating this procedure eventually gives

$$\tfrac{1}{(l+1)!}m(t_{l+1}-a)^{l+1} \le f(t_{l+1}) - T_l(t_{l+1}) \le \tfrac{1}{(l+1)!}M(t_{l+1}-a)^{l+1}$$

for all $t_{l+1} > a$ in I. In particular $\frac{1}{(l+1)!}m(x-a)^{l+1} \le f(x) - T_l(x) \le \frac{1}{(l+1)!}M(x-a)^{l+1}$, so

$$m \le \big(f(x) - T_l(x)\big)\tfrac{(l+1)!}{(x-a)^{l+1}} \le M\,.$$

By the Intermediate Value Theorem, there is a number $c \in [a, x]$ such that

$$f^{(l+1)}(c) = \big(f(x) - T_l(x)\big)\frac{(l+1)!}{(x-a)^{l+1}}$$

.

9.8 Theorem: *The functions e^x, $\sin x$, $\cos x$ and $(1+x)^p$ are all exactly equal to the sum of their Taylor series centred at 0 in the interval of convergence.*

Proof: First let $f(x) = e^x$ and let $x \in \mathbf{R}$. By Taylor's Theorem, $f(x) - T_l(x) = \dfrac{e^c x^{l+1}}{(l+1)!}$ for some c between 0 and x, and so

$$\left|f(x) - T_l(x)\right| \leq \frac{e^{|x|}|x|^{l+1}}{(l+1)!}\,.$$

Since $\sum \dfrac{e^{|x|}|x|^{l+1}}{(l+1)!}$ converges by the Ratio Test, we have $\lim\limits_{l\to\infty} \dfrac{e^{|x|}|x|^{l+1}}{(l+1)!} = 0$ by the Divergence Test, so $\lim\limits_{l\to\infty}\big(f(x) - T_l(x)\big) = 0$, and so $f(x) = \lim\limits_{l\to\infty} T_l(x) = T(x)$.

Now let $f(x) = \sin x$ and let $x \in \mathbf{R}$. By Taylor's Theorem, $f(x) - T(x) = \dfrac{f^{(l+1)}(c)\,x^{l+1}}{(l+1)!}$ for some c between 0 and x. Since $f^{(l+1)}(x)$ is one of the functions $\pm\sin x$ or $\pm\cos x$, we have $\left|f^{(l+1)}(c)\right| \leq 1$ for all c and so

$$\left|f(x) - T(x)\right| \leq \frac{|x|^{l+1}}{(l+1)!}\,.$$

Since $\sum \dfrac{|x|^{l+1}}{(l+1)!}$ converges by the Ratio Test, $\lim\limits_{l\to\infty} \dfrac{|x|^{l+1}}{(l+1)!} = 0$ by the Divergence Test, and so we have and $f(x) = T(x)$ as above.

Let $f(x) = \cos x$. For all $x \in \mathbf{R}$ we have

$$\begin{aligned} f(x) = \cos x &= \tfrac{d}{dx}\sin x \\ &= \tfrac{d}{dx}\left(x - \tfrac{1}{3!}\,x^3 + \tfrac{1}{5!}\,x^5 - \tfrac{1}{7!}\,x^7 + \cdots\right) \\ &= 1 - \tfrac{1}{2!}\,x^2 + \tfrac{1}{4!}\,x^4 - \tfrac{1}{6!}\,x^6 + \cdots \end{aligned}$$

which is the sum of its Taylor series, centred at 0.

Finally, let $f(x) = (1+x)^p$. The Taylor series centred at 0 is

$$T(x) = 1 + px + \tfrac{p(p-1)}{2!}x^2 + \tfrac{p(p-1)(p-2)}{3!}x^3 + \tfrac{p(p-1)(p-2)(p-3)}{4!}x^4 + \cdots$$

and it converges for $|x| < 1$. Differentiating the power series gives

$$T'(x) = p + \tfrac{p(p-1)}{1!}x + \tfrac{p(p-1)(p-2)}{2!}x^2 + \tfrac{p(p-1)(p-2)(p-3)}{3!}x^3 + \cdots$$

and so

$$\begin{aligned} (1+x)T'(x) &= p + \left(p + \tfrac{p(p-1)}{1!}\right)x + \left(\tfrac{p(p-1)}{1!} + \tfrac{p(p-1)(p-2)}{2!}\right)x^2 \\ &\qquad + \left(\tfrac{p(p-1)(p-2)}{2!} - \tfrac{p(p-1)(p-2)(p-3)}{3!}\right)x^3 + \cdots \\ &= p + \tfrac{p\cdot p}{1!}x + \tfrac{p\cdot p(p-1)}{2!}x^2 + \tfrac{p\cdot p(p-1)(p-2)}{3!}x^3 + \cdots \\ &= p\,T(x)\,. \end{aligned}$$

Thus we have $(1+x)T'(x) = pT(x)$ with $T(0) = 1$. This DE is linear since we can write it as $T'(x) - \frac{p}{1+x}T(x) = 0$. An integrating factor is $\lambda = e^{\int -\frac{p}{1+x}\,dx} = e^{-p\ln(1+x)} = (1+x)^{-p}$ and the solution is $T(x) = (1+x)^{-p}\displaystyle\int 0\,dx = b(1+x)^p$ for some constant b. Since $T(0) = 1$ we have $b = 1$ and so $T(x) = (1+x)^p = f(x)$.

10. Applications

10.1 Example: Let $f(x) = \sin\left(\frac{1}{2}x^2\right)$. Find the 10^{th} derivative $f^{(10)}(0)$.

Solution: We have

$$\begin{aligned} f(x) &= \sin\left(\tfrac{1}{2}\,x^2\right) \\ &= \left(\tfrac{1}{2}\,x^2\right) - \tfrac{1}{3!}\left(\tfrac{1}{2}\,x^2\right)^3 + \tfrac{1}{5!}\left(\tfrac{1}{2}\,x^2\right)^5 - \cdots \\ &= \tfrac{1}{2}\,x^2 - \tfrac{1}{2^3\,3!}\,x^6 + \tfrac{1}{2^5\,5!}\,x^{10} - \cdots \end{aligned}$$

We have $c_{10} = \frac{1}{2^5\,5!}$ and so $f^{(10)}(0) = 10!\,c_{10} = \frac{10!}{2^5\,5!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{2^5} = 5\cdot 9\cdot 7\cdot 3 = 945\,.$

10.2 Example: Find $\displaystyle\lim_{x\to 0}\frac{e^{-2x^2} - \cos 2x}{\big(\tan^{-1}x - \ln(1+x)\big)^2}$

Solution: We have

$$\begin{aligned} \frac{e^{-2x^2} - \cos 2x}{\big(\tan^{-1}x - \ln(1+x)\big)^2} &= \frac{\left(1 - (2x^2) + \frac{1}{2!}(2x^2)^2 - \cdots\right) - \left(1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \cdots\right)}{\left(\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots\right) - \left(x - \frac{1}{2}x^3 + \frac{1}{3}x^3 - \cdots\right)\right)^2} \\ &= \frac{\left(1 - 2x^2 + 2x^4 - \cdots\right) - \left(1 - 2x^2 + \frac{2}{3}x^4 - \cdots\right)}{\left(\frac{1}{2}x^2 - \frac{2}{3}x^3 + \cdots\right)^2} \\ &= \frac{\frac{4}{3}x^4 + \cdots}{\frac{1}{4}x^4 + \cdots} = \tfrac{1}{3} + c_1 x + \cdots \longrightarrow \tfrac{1}{3} \text{ as } x \to 0\,. \end{aligned}$$

10.3 Example: Approximate the value of $\frac{1}{\sqrt{e}}$ so the error is at most $\frac{1}{100}$.

Solution: We have

$$\begin{aligned} \tfrac{1}{\sqrt{e}} = e^{-1/2} &= 1 - \left(\tfrac{1}{2}\right) + \tfrac{1}{2!}\left(\tfrac{1}{2}\right)^2 - \tfrac{1}{3!}\left(\tfrac{1}{2}\right)^3 + \tfrac{1}{4!}\left(\tfrac{1}{2}\right)^4 - \cdots \\ &= 1 - \tfrac{1}{2} + \tfrac{1}{2^2\,2!} - \tfrac{1}{2^3\,3!} + \tfrac{1}{2^4\,4!} - \cdots \\ &\cong 1 - \tfrac{1}{2} + \tfrac{1}{2^2\,2!} - \tfrac{1}{2^3\,3!} = 1 - \tfrac{1}{2} + \tfrac{1}{8} - \tfrac{1}{48} = \tfrac{29}{48} \end{aligned}$$

with absolute error $E \le \frac{1}{2^4\,4!} = \frac{1}{384}$, by the Alternating Series Test.

10.4 Example: Approximate the value of $\sqrt{e}$ so the error is at most $\frac{1}{100}$.

Solution: We have

$$\begin{aligned} \sqrt{e} = e^{1/2} &= 1 + \left(\tfrac{1}{2}\right) + \tfrac{1}{2!}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{3!}\left(\tfrac{1}{2}\right)^3 + \tfrac{1}{4!}\left(\tfrac{1}{2}\right)^4 + \tfrac{1}{5!}\left(\tfrac{1}{2}\right)^5 + \cdots \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2^2\,2!} + \tfrac{1}{2^3\,3!} + \tfrac{1}{2^4\,4!} + \tfrac{1}{2^5\,5!} + \cdots \\ &\cong 1 + \tfrac{1}{2} + \tfrac{1}{2^2\,2!} + \tfrac{1}{2^3\,3!} = 1 + \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{48} = \tfrac{79}{48} \end{aligned}$$

with absolute error

$$\begin{aligned} E &= \tfrac{1}{2^4\,4!} + \tfrac{1}{2^5\,5!} + \tfrac{1}{2^6\,6!} + \tfrac{1}{2^7\,7!} + \tfrac{1}{2^8\,8!} + \cdots \\ &= \tfrac{1}{2^4\,4!}\left(\tfrac{1}{2\cdot 5} + \tfrac{1}{2^2\cdot 6\cdot 5} + \tfrac{1}{2^3\cdot 7\cdot 6\cdot 5} + \tfrac{1}{2^4\cdot 8\cdot 7\cdot 6\cdot 5} + \cdots\right) \\ &\le \tfrac{1}{2^4\,4!}\left(\tfrac{1}{2\cdot 5} + \tfrac{1}{2^2 5^2} + \tfrac{1}{2^3 5^3} + \tfrac{1}{2^4 5^4} + \cdots\right) \\ &= \tfrac{1}{384}\cdot\frac{1}{1 - \frac{1}{10}} = \tfrac{1}{384}\cdot\tfrac{10}{9} = \tfrac{5}{1728} < \tfrac{1}{100}\,, \end{aligned}$$

where we used the Comparison Test and the formula for the sum of a geometric series.

10.5 Example: Approximate the value of $\ln 2$ so the error is at most $\frac{1}{50}$

Solution: We provide two solutions. For both solutions, we use the fundtion

$$f(x) = \ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots$$

For the first solution, we put in $x = 1$ to get

$$\begin{aligned} \ln 2 = f(1) &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots \\ &\cong 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots + \tfrac{1}{49} \end{aligned}$$

with absolute error $E \leq \frac{1}{50}$ by the Alternating Series Test. It would be cumbersome to add up the 49 terms in the above alternating sum, so we provide a second solution in which we put in $x = -\frac{1}{2}$. We have

$$\begin{aligned} \ln 2 = -\ln\tfrac{1}{2} = -f\big(-\tfrac{1}{2}\big) &= -\left(-\tfrac{1}{2}\right) + \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^2 - \tfrac{1}{3}\left(-\tfrac{1}{2}\right)^3 + \tfrac{1}{4}\left(-\tfrac{1}{2}\right)^4 - \cdots \\ &= \tfrac{1}{2} + \tfrac{1}{2\cdot 2^2} + \tfrac{1}{3\cdot 2^3} + \tfrac{1}{4\cdot 2^4} + \tfrac{1}{5\cdot 2^5} + \tfrac{1}{6\cdot 2^6} + \cdots \\ &\cong \tfrac{1}{2} + \tfrac{1}{2\cdot 2^2} + \tfrac{1}{3\cdot 2^3} + \tfrac{1}{4\cdot 2^4} = \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{24} + \tfrac{1}{64} = \tfrac{131}{192} \end{aligned}$$

with absolute error

$$\begin{aligned} E &= \tfrac{1}{5\cdot 2^5} + \tfrac{1}{6\cdot 2^6} + \tfrac{1}{7\cdot 2^7} + \tfrac{1}{8\cdot 2^8} + \cdots \\ &\leq \tfrac{1}{5\cdot 2^5} + \tfrac{1}{5\cdot 2^6} + \tfrac{1}{5\cdot 2^7} + \tfrac{1}{5\cdot 2^8} + \cdots \\ &= \frac{\frac{1}{5\cdot 2^5}}{1 - \frac{1}{2}} = \tfrac{2}{5\cdot 2^5} = \tfrac{1}{80} \end{aligned}$$

by the Comparison Test and the formula for the sum of a geometric series.

10.6 Example: Approximate the value of $10^{2/3}$ so the error is at most $\frac{1}{100}$.

Solution: We use the function

$$f(x) = (1+x)^{2/3} = 1 + \frac{\left(\frac{2}{3}\right)}{1!}\,x \frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{2!}\,x^2 + \frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}\,x^3 + \frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{4!}\,x^4 + \cdots$$

We have

$$\begin{aligned} 10^{2/3} &= (8+2)^{2/3} = 4\big(1 + \tfrac{1}{4}\big)^{2/3} = 4\,f\left(\tfrac{1}{4}\right) \\ &= 4\left(1 + \frac{\left(\frac{2}{3}\right)}{4\cdot 1!} + \frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{4^2\cdot 2!} + \frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{4^3\cdot 3!} + \frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{4^4\cdot 4!} + \cdots\right) \\ &= 4 + \tfrac{8}{12\cdot 1!} - \tfrac{8\cdot 1}{12^2\cdot 2!} + \tfrac{8\cdot 1\cdot 4}{12^3\cdot 3!} - \tfrac{8\cdot 1\cdot 4\cdot 7}{12^4\cdot 4!} + \cdots \\ &\cong 4 + \tfrac{8}{12\cdot 1!} - \tfrac{8\cdot 1}{12^2\cdot 2!} = 4 + \tfrac{2}{3} - \tfrac{1}{36} = \tfrac{167}{36} \end{aligned}$$

with absolute error $E \leq \frac{8\cdot 1\cdot 4}{12^3\cdot 3!} = \frac{1}{324}$ by the Alternating Series Test.

10.7 Example: Approximate the value of π so the error is at most $\frac{1}{50}$.

Solution: We provide two solutions. For both solutions we use the function

$$f(x) = \tan^{-1} x = x - \tfrac{1}{3}\,x^3 + \tfrac{1}{5}\,x^5 - \tfrac{1}{7}\,x^7 + \cdots$$

For the first solution, we put in $x = 1$ to get

$$\pi = 4 \cdot \tfrac{\pi}{4} = 4f(1) = 4\big(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots\big) \cong 4\big(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots + \tfrac{1}{399}\big)$$

with absolute error $E \le \frac{4}{201}$ by the Alternating Series Test. It would be cumbersome to add up the 100 terms in the alternating sum, so we provide a second solution in which we put in $x = \frac{1}{\sqrt{3}}$. We have

$$\begin{aligned}\pi = 6 \cdot \tfrac{\pi}{6} = 6f\big(\tfrac{1}{\sqrt{3}}\big) &= 6\Big(\tfrac{1}{\sqrt{3}} - \tfrac{1}{3\cdot\sqrt{3}^{\,3}} + \tfrac{1}{5\cdot\sqrt{3}^{\,5}} - \tfrac{1}{7\cdot\sqrt{3}^{\,7}} + \tfrac{1}{9\cdot\sqrt{3}^{\,9}} - \cdots\Big)\\ &= 2\sqrt{3}\left(1 - \tfrac{1}{3\cdot3} + \tfrac{1}{5\cdot3^2} - \tfrac{1}{7\cdot3^3} + \tfrac{1}{9\cdot3^4} - \cdots\right)\\ &\cong 2\sqrt{3}\left(1 - \tfrac{1}{3\cdot3} + \tfrac{1}{5\cdot3^2}\right) = \tfrac{82\sqrt{3}}{45}\end{aligned}$$

with absolute error $E \le \frac{2\sqrt{3}}{7\cdot3^3} = \frac{2\sqrt{3}}{189}$ by the Alternating Series Test. We remark that in order to make this approximation, we must first approximate $\sqrt{3}$.

10.8 Example: Approximate the value of $\sin\big(10^\circ\big)$ so the error is at most $\frac{1}{1000}$.

Solution: We use the function

$$f(x) = \sin x = x - \tfrac{1}{3!}\,x^3 + \tfrac{1}{5!}\,x^5 - \cdots$$

We put in $x = 10^\circ = \frac{\pi}{18}$ to get

$$\sin(10^\circ) = f\left(\tfrac{\pi}{18}\right) = \tfrac{\pi}{18} - \tfrac{1}{3!}\left(\tfrac{\pi}{18}\right)^3 + \tfrac{1}{5!}\left(\tfrac{\pi}{18}\right)^5 - \cdots \cong \tfrac{\pi}{18}$$

with absolute error $E \le \frac{1}{3!}\left(\frac{\pi}{18}\right)^3$ by the Alternating Series Test. We remark that in order to make this approximation, we must first approximate π.

10.9 Example: Approximate the value of $\displaystyle\int_0^1 e^{-x^2}\,dx$ so the error is at most $\frac{1}{100}$.

Solution: We have

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &= \int_0^1 \Big(1 - x^2 + \tfrac{1}{2!}\,x^4 - \tfrac{1}{3!}\,x^6 + \tfrac{1}{4!}\,x^8 - \cdots\Big)\,dx\\ &= \Big[x - \tfrac{1}{3}\,x^3 + \tfrac{1}{5\cdot2!}\,x^5 - \tfrac{1}{7\cdot3!}\,x^7 + \tfrac{1}{9\cdot4!}\,x^9 - \cdots\Big]_0^1\\ &= 1 - \tfrac{1}{3} + \tfrac{1}{5\cdot2!} - \tfrac{1}{7\cdot3!} + \tfrac{1}{9\cdot4!} - \cdots\\ &\cong 1 - \tfrac{1}{3} + \tfrac{1}{5\cdot2!} - \tfrac{1}{7\cdot3!} = \tfrac{26}{35}\end{aligned}$$

with absolute error $E \le \frac{1}{9\cdot4!} = \frac{1}{216}$ by the Alternating Series Test.

10.10 Example: Approximate the value of $\displaystyle\int_0^{\sqrt{2}} \frac{\sin x}{x}\,dx$ so the error is at most $\frac{1}{50}$.

10.11 Example: Find the exact value of the sum $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{(2n)!}$.

Solution: We have

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{2}^{\,2n}}{(2n)!} = \cos(\sqrt{2})\,.$$

10.12 Example: Find the exact value of the sum $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{(-3)^n}$.

Solution: Note first that

$$\sum_{n=1}^{\infty} \frac{n-2}{(-3)^n} = \sum_{n=1}^{\infty} \frac{n}{(-3)^n} - \sum_{n=1}^{\infty} \frac{2}{(-3)^n}\,.$$

The second sum on the right is geometric with first term $-\frac{2}{3}$ and ratio $-\frac{1}{3}$, so we have

$$\sum_{n=1} \frac{2}{(-3)^n} = \frac{-\frac{2}{3}}{1+\frac{1}{3}} = -\tfrac{1}{2}\,.$$

To find the first sum on the right, we begin with the fact that for $|x|<1$ we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Differentiate both sides to get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^3 + 4x^3 + \cdots$$

Multiply both sides by x to get

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

Thus we obtain the formula

$$\sum_{n=1}^{\infty} n\,x^n = \frac{x}{(1-x)^2} \text{ for all } |x|<1\,.$$

Put in $x = -\frac{1}{3}$ to get

$$\sum_{n=1}^{\infty} \frac{n}{(-3)^n} = \frac{-\frac{1}{3}}{\left(1+\frac{1}{3}\right)^2} = -\tfrac{3}{16}\,.$$

Thus we have

$$\sum_{n=1}^{\infty} \frac{n}{(-3)^n} = \sum_{n=1}^{\infty} \frac{n}{(-3)^n} - \sum_{n=1}^{\infty} \frac{2}{(-3)^n} = -\tfrac{3}{16} + \tfrac{1}{2} = \tfrac{5}{16}\,.$$

10.13 Example: Find the exact value of the sum $\displaystyle\sum_{n=0}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}{5^n\, n!}$.

Solution: We have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}{5^n\, n!} &= 2 \sum_{n=0}^{\infty} \frac{\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)\left(\frac{11}{3}\right)\cdots\left(\frac{3n+2}{3}\right)}{n!} \cdot \frac{3^n}{5^n} \\
&= 2 \sum_{n=0}^{\infty} \frac{\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)\cdots\left(-\frac{3n+2}{3}\right)}{n!} \cdot \left(-\frac{3}{5}\right)^n \\
&= 2\left(1 - \frac{3}{5}\right)^{-5/3} = 2 \cdot \left(\frac{5}{2}\right)^{5/3} .
\end{aligned}$$