Chapter 3. The Riemann Integral

The Riemann Integral

3.1 Definition: A **partition** of the closed interval $[a, b]$ is a set $X = \{x_0, x_1, \cdots, x_n\}$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b\,.$$

The intervals $[x_{i-1}, x_i]$ are called the **subintervals** of $[a, b]$, and we write

$$\Delta_i x = x_i - x_{i-1}$$

for the size of the i^{th} subinterval. Note that

$$\sum_{i=1}^{n} \Delta_i x = b - a\,.$$

The **size** of the partition X, denoted by $|X|$ is

$$|X| = \max\left\{\Delta_i x \middle| 1 \leq i \leq n\right\}.$$

3.2 Definition: Let X be a partition of $[a, b]$, and let $f : [a, b] \to \mathbb{R}$ be bounded. A **Riemann sum** for f on X is a sum of the form

$$S = \sum_{i=1}^{n} f(t_i)\Delta_i x \qquad \text{for some } t_i \in [x_{i-1}, x_i]\,.$$

The points t_i are called **sample points**.

3.3 Definition: Let $f : [a, b] \to \mathbb{R}$ be bounded. We say that f is **(Riemann) integrable** on $[a, b]$ when there exists a number I with the property that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every partition X of $[a, b]$ with $|X| < \delta$ we have $|S - I| < \epsilon$ for every Riemann sum for f on X, that is

$$\left|\sum_{i=1}^{n} f(t_i)\Delta_i x - I\right| < \epsilon\,.$$

for every choice of $t_i \in [x_{i-1}, x_i]$ This number I is unique (as we prove below); it is called the **(Riemann) integral** of f on $[a, b]$, and we write

$$I = \int_a^b f \ \text{, or } I = \int_a^b f(x)\,dx\,.$$

Proof: Suppose that I and J are two such numbers. Let $\epsilon > 0$ be arbitrary. Choose δ_1 so that for every partition X with $|X| < \delta_1$ we have $|S - I| < \frac{\epsilon}{2}$ for every Riemann sum S on X, and choose $\delta_2 > 0$ so that for every partition X with $|X| < \delta_2$ we have $|S - J| < \frac{\epsilon}{2}$ for every Riemann sum S on X. Let $\delta = \min\{\delta_1, \delta_2\}$. Let X be any partition of $[a, b]$ with $|X| < \delta$. Choose $t_i \in [x_{i-1}, x_i]$ and let $S = \sum\limits_{i=1}^{n} f(t_i)\Delta_i x$. Then we have $|I - J| \leq |I - S| + |S - J| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ was arbitrary, we must have $I = J$.

3.4 Example: Let $f(x)=\begin{cases}1 & \text{if} \quad x\in\mathbb{Q}\\ 0 & \text{if} \quad x\notin\mathbb{Q}\,.\end{cases}$ Show that f is not integrable on $[0,1]$.

Solution: Suppose, for a contradiction, that f is integrable on $[0,1]$, and write $I=\int_0^1 f$. Let $\epsilon=\frac{1}{2}$. Choose δ so that for every partition X with $|X|<\delta$ we have $|S-I|<\frac{1}{2}$ for every Riemann sum S for f on X. Choose a partition X with $|X|<\delta$. Let $S_1=\sum\limits_{i=1}^n f(t_i)\Delta_i x$ where each $t_i\in[x_{i-1},x_i]$ is chosen with $t_i\in\mathbb{Q}$, and let $S_2=\sum\limits_{i=1}^n f(s_i)\Delta_i x$ where each $s_i\in[x_{i-1},x_i]$ is chosen with $s_i\notin\mathbb{Q}$. Note that we have $|S_1-I|<\frac{1}{2}$ and $|S_2-I|<\frac{1}{2}$. Since each $t_i\in\mathbb{Q}$ we have $f(t_i)=1$ and so $S_1=\sum\limits_{i=1}^n f(t_i)\Delta_i x=\sum\limits_{i=1}^n \Delta_i x=1-0=1$, and since each $s_i\notin\mathbb{Q}$ we have $f(s_i)=0$ and so $S_2=\sum\limits_{i=1}^n f(s_i)\Delta_i x=0$. Since $|S_1-I|<\frac{1}{2}$ we have $|1-I|<\frac{1}{2}$ and so $\frac{1}{2}<I<\frac{3}{2}$, and since $|S_2-I|<\frac{1}{2}$ we have $|0-I|<\frac{1}{2}$ and so $-\frac{1}{2}<I<\frac{1}{2}$, giving a contradiction.

3.5 Example: Show that the constant function $f(x)=c$ is integrable on any interval $[a,b]$ and we have $\displaystyle\int_a^b c\,dx=c(b-a)$.

Solution: The solution is left as an exercise.

3.6 Example: Show that the identity function $f(x)=x$ is integrable on any interval $[a,b]$, and we have $\displaystyle\int_a^b x\,dx=\frac{1}{2}(b^2-a^2)$.

Solution: Let $\epsilon>0$. Choose $\delta=\frac{2\epsilon}{b-a}$. Let X be any partition of $[a,b]$ with $|X|<\delta$. Let $t_i\in[x_{i-1},x_i]$ and set $S=\sum\limits_{i=1}^n f(t_i)\Delta_i x=\sum\limits_{i=1}^n t_i\Delta_i x$. We must show that $|S-\frac{1}{2}(b^2-a^2)|<\epsilon$. Notice that

$$\begin{aligned}
\sum_{i=1}^n (x_i+x_{i-1})\Delta_i x &= \sum_{i=1}^n (x_i+x_{i-1})(x_i-x_{i-1}) = \sum_{i=1}^n {x_i}^2-{x_{i-1}}^2\\
&= ({x_1}^2-{x_0}^2)+({x_2}^2-{x_1}^2)+\cdots+({x_{n-1}}^2-{x_{n-2}}^2)+({x_n}^2-{x_{n-1}}^2)\\
&= -{x_0}^2+({x_1}^2-{x_1}^2)+\cdots+({x_{n-1}}^2-{x_{n-1}}^2)+{x_n}^2\\
&= {x_n}^2-{x_0}^2=b^2-a^2
\end{aligned}$$

and that when $t_i\in[x_{i-1},x_i]$ we have $\left|t_i-\frac{1}{2}(x_i+x_{i-1})\right|\le\frac{1}{2}(x_i-x_{i-1})=\frac{1}{2}\Delta_i x$, and so

$$\begin{aligned}
\left|S-\tfrac{1}{2}(b^2-a^2)\right| &= \Big|\sum_{i=1}^n t_i\Delta_i x-\tfrac{1}{2}\sum_{i=1}^n (x_i+x_{i-1})\Delta_i x\Big|\\
&= \Big|\sum_{i=1}^n \left(t_i-\tfrac{1}{2}(x_i+x_{i+1})\right)\Delta_i x\Big|\\
&\le \sum_{i=1}^n \left|t_i-\tfrac{1}{2}(x_i+x_{i+1})\right|\Delta_i x\\
&\le \sum_{i=1}^n \tfrac{1}{2}\Delta_i x\Delta_i x\le\sum_{i=1}^n \tfrac{1}{2}\delta\Delta_i x\\
&= \tfrac{1}{2}\delta(b-a)=\epsilon\,.
\end{aligned}$$

Upper and Lower Riemann Sums

3.7 Definition: Let X be a partition for $[a,b]$ and let $f:[a,b]\to\mathbb{R}$ be bounded. The **upper Riemann sum** for f on X, denoted by $U(f,X)$, is

$$U(f,X)=\sum_{i=1}^{n} M_i\,\Delta_i x \quad \text{ where } M_i=\sup\left\{f(t)\middle|t\in[x_{i-1},x_i]\right\}$$

and the **lower Riemann sum** for f on X, denoted by $L(f,X)$ is

$$L(f,X)=\sum_{i=1}^{n} m_i\,\Delta_i x \quad \text{ where } m_i=\inf\left\{f(t)\middle|t\in[x_{i-1},x_i]\right\}.$$

3.8 Remark: The upper and lower Riemann sums $U(f,X)$ and $L(f,X)$ are not, in general, Riemann sums at all, since we do not always have $M_i=f(t_i)$ or $m_i=f(s_i)$ for any $t_i,s_i\in[x_{i-1},x_i]$. If f is increasing, then $M_i=f(x_i)$ and $m_i=f(x_{i-1})$, and so in this case $U(f,X)$ and $L(f,X)$ are indeed Riemann sums. Similarly, if f is decreasing then $U(f,X)$ and $L(f,X)$ are Riemann sums. Also, if f is continuous then, by the Extreme Value Theorem, we have $M_i=f(t_i)$ and $m_i=f(s_i)$ for some $t_i,s_i\in[x_{i-1},x_i]$, and so in this case $U(f,X)$ and $L(f,X)$ are again Riemann sums.

3.9 Note: Let X be a partition of $[a,b]$, and let $f:[a,b]\to\mathbb{R}$. be bounded. Then

$$\begin{aligned} U(f,X)&=\sup\left\{S\middle|S \text{ is a Riemann sum for } f \text{ on } X\right\}\text{ , and}\\ L(f,X)&=\inf\left\{S\middle|S \text{ is a Riemann sum for } f \text{ on } X\right\}. \end{aligned}$$

In particular, for every Riemann sum S for f on X we have

$$L(f,X)\le S\le U(f,X)$$

Proof: We show that $U(f,X)=\sup\left\{S\middle|S \text{ is a Riemann sum for } f \text{ on } X\right\}$ (the other statement is proved similarly). Let $\mathcal{T}=\left\{S\middle|S \text{ is a Riemann sum for } f \text{ on } X\right\}$. For $S\in\mathcal{T}$, say $S=\sum\limits_{i=1}^{n} f(t_i)\Delta_i x$ where $t_i\in[x_{i-1},x_i]$, we have

$$S=\sum_{i=1}^{n} f(t_i)\Delta_i x\le\sum_{i=1}^{n} M_i\Delta_i x=U(f,X).$$

Thus $U(f,X)$ is an upper bound for $\mathcal{T}$ so we have $U(f,X)\ge\sup\mathcal{T}$. It remains to show that given any $\epsilon>0$ we can find $S\in\mathcal{T}$ with $U(f,X)-S<\epsilon$. Let $\epsilon>0$ be arbitrary. Since $M_i=\sup\left\{f(t)\middle|t\in[x_{i-1},x_i]\right\}$, we can choose $t_i\in[x_{i-1},x_i]$ with $M_i-f(t_i)<\frac{\epsilon}{b-a}$. Then we have

$$U(f,X)-S=\sum_{i=1}^{n} M_i\Delta_i x-\sum_{i=1}^{n} f(t_i)\Delta_i x=\sum_{i=1}^{n}\big(M_i-f(t_i)\big)\Delta_i x<\sum_{i=1}^{n}\tfrac{\epsilon}{b-a}\,\Delta_i x=\epsilon$$

3.10 Lemma: *Let $f:[a,b]\to\mathbb{R}$ be bounded with upper and lower bounds M and m. Let X and Y be partitions of $[a,b]$ such that $Y=X\cup\{c\}$ for some $c\notin X$. Then*

$$0\leq L(f,Y)-L(f,X)\leq (M-m)|X|\ ,\ \text{and}$$
$$0\leq U(f,X)-U(f,Y)\leq (M-m)|X|\,.$$

Proof: We shall prove that $0\leq L(f,Y)-L(f,X)\leq (M-m)|X|$ (the proof that $0\leq U(f,X)-U(f,Y)\leq (M-m)|X|$ is similar). Say $X=\{x_0,x_1,\cdots,x_n\}$ and $c\in[x_{i-1},x_i]$ so $Y=\{x_0,x_1,\cdots,x_{i-1},c,x_i,\cdots,x_n\}$. Then

$$L(f,Y)-L(f,X)=k_i(c-x_{i-1})+l_i(x_i-c)-m_i(x_i-x_{i-1})$$

where

$$k_i=\inf\big\{f(t)\big|t\in[x_{i-1},c]\big\},\ l_i=\inf\big\{f(t)\big|t\in[c,x_i]\big\}\ ,\ m_i=\inf\big\{f(t)\big|t\in[x_{i-1},x_i]\big\}\,.$$

Since $m_i=\min\{k_i,l_i\}$ we have $k_i\geq m_i$ and $l_i\geq m_i$, so

$$L(f,Y)-L(f,X)\geq m_i(c-x_{i-1})+m_i(x_i-c)-m_i(x_i-x_{i-1})=0\,.$$

Since $k_i\leq M$ and $l_i\leq M$ and $m_i\geq m$ we have

$$\begin{aligned}L(f,Y)-L(f,X)&\leq M(c-x_{i-1})+M(x_i-c)-m(x_i-x_{i-1})\\&=(M-m)(x_i-x_{i-1})\leq (M-m)|X|\,.\end{aligned}$$

3.11 Note: Let X and Y be partitions of $[a,b]$ with $X\subset Y$. Then

$$L(f,X)\leq L(f,Y)\leq U(f,Y)\leq U(f,X)\,.$$

Proof: If Y is obtained by adding one point to X then this follows from the above lemma. In general, Y can be obtained by adding finitely many points to X, one point at a time.

3.12 Note: Let X and Y be any partitions of $[a,b]$. Then $L(f,X)\leq U(f,Y)$.

Proof: Let $Z=X\cup Y$. Then by the above note,

$$L(f,X)\leq L(f,Z)\leq U(f,Z)\leq U(f,Y)\,.$$

3.13 Definition: Let $f:[a,b]\to\mathbb{R}$ be bounded. The **upper integral** of f on $[a,b]$, denoted by $U(f)$, is given by

$$U(f)=\inf\big\{U(f,X)\big|X\text{ is a partition of }[a,b]\big\}$$

and the **lower integral** of f on $[a,b]$, denoted by $L(f)$, is given by

$$L(f)=\sup\big\{L(f,X)\big|X\text{ is a partition of }[a,b]\big\}\,.$$

3.14 Note: The upper and lower integrals of f both exist even when f is not integrable.

3.15 Note: We always have $L(f)\leq U(f)$.

Proof: Let $\epsilon>0$ be arbitrary. Choose a partition X_1 so that $L(f)-L(f,X_1)<\frac{\epsilon}{2}$ and choose a partition X_2 so that $U(f,X_2)-U(f)<\frac{\epsilon}{2}$. Then

$$\begin{aligned}U(f)-L(f)&=\big(U(f)-U(f,X_2)\big)+\big(U(f,X_2)-L(f,X_1)\big)+\big(L(f,X_1)-L(f)\big)\\&>-\tfrac{\epsilon}{2}+0-\tfrac{\epsilon}{2}=-\epsilon\,.\end{aligned}$$

Since ϵ was arbitrary, this implies that $U(f)-L(f)\geq 0$.

3.16 Theorem: *(Equivalent Definitions of Integrability) Let $f : [a, b] \to \mathbb{R}$ be bounded. Then the following are equivalent.*
(1) f is integrable on $[a, b]$.
(2) For all $\epsilon > 0$ there exists a partition X such that $U(f, X) - L(f, X) < \epsilon$.
(3) $L(f) = U(f)$.

Proof: (1) $\Longrightarrow$ (2). Suppose that f is integrable on $[a, b]$ with $I = \int_a^b f$. Let $\epsilon > 0$. Choose $\delta > 0$ so that for every partition X with $|X| < \delta$ we have $|S - I| < \frac{\epsilon}{4}$ for every Riemann sum S on X. Let X be a partition with $|X| < \delta$. Let S_1 be a Riemann sum for f on X with $|U(f, X) - S_1| < \frac{\epsilon}{4}$, and let S_2 be a Riemann sum for f on X with $|S_2 - L(f, X)| < \frac{\epsilon}{4}$. Then

$$\begin{aligned} |U(f, X) - L(f, X)| &\leq |U(f, X) - S_1| + |S_1 - I| + |I - S_2| + |S_2 - L(f, X)| \\ &< \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} = \epsilon\,. \end{aligned}$$

(2)$\Longrightarrow$(3). Suppose that for all $\epsilon > 0$ there is a partition X such that $U(f, X) - L(f, X) < \epsilon$. Let $\epsilon > 0$. Choose X so that $U(f, X) - L(f, X) < \epsilon$. Then

$$\begin{aligned} U(f) - L(f) &= \big(U(f) - U(f, X)\big) + \big(U(f, X) - L(f, X)\big) + \big(L(f, X) - L(f)\big) \\ &< 0 + \epsilon + 0 = \epsilon\,. \end{aligned}$$

Since $0 \leq U(f) - L(f) < \epsilon$ for every $\epsilon > 0$, we have $U(f) = L(f)$.

(3) $\Longrightarrow$ (1). Suppose that $L(f) = U(f)$ and let $I = L(f) = U(f)$. Let $\epsilon > 0$. Choose a partition X_0 of $[a, b]$ so that $L(f) - L(f, X_0) < \frac{\epsilon}{2}$ and $U(f, X_0) - U(f) < \frac{\epsilon}{2}$. Say $X_0 = \{x_0, x_1, \cdots, x_n\}$ and set $\delta = \frac{\epsilon}{2(n-1)(M-m)}$, where M and m are upper and lower bounds for f on $[a, b]$. Let X be any partition of $[a, b]$ with $|X| < \delta$. Let $Y = X_0 \cup X$. Note that Y is obtained from X by adding at most $n - 1$ points, and each time we add a point, the size of the new partition is at most $|X| < \delta$. By lemma 3.10, applied $n - 1$ times, we have

$$\begin{aligned} &0 \leq U(f, X) - U(f, Y) \leq (n - 1)(M - m)|X| < (n - 1)(M - m)\delta = \tfrac{\epsilon}{2}\ , \text{ and} \\ &0 \leq L(f, Y) - L(f, X) \leq (n - 1)(M - m)|X| < (n - 1)(M - m)\delta = \tfrac{\epsilon}{2}\,. \end{aligned}$$

Now let S be any Riemann sum for f on X. Note that $L(f, X_0) \leq L(f, Y) \leq L(f) = U(f) \leq U(f, Y) \leq U(f, X_0)$ and $L(f, X) \leq S \leq U(f, X)$, so we have

$$\begin{aligned} S - I \leq U(f, X) - I = U(f, X) - U(f) &= \big(U(f, X) - U(f, Y)\big) + \big(U(f, Y) - U(f)\big) \\ &\leq \big(U(f, X) - U(f, Y)\big) + \big(U(f, X_0) - U(f)\big) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \end{aligned}$$

and

$$\begin{aligned} I - S = I - L(f, X) = L(f) - L(f, X) &= \big(L(f) - L(f, Y)\big) + \big(L(f, Y) - L(f, X)\big) \\ &\leq \big(L(f) - L(f, X_0)\big) + \big(L(f, Y) - L(f, X)\big) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

Evaluating Integrals of Continuous Functions

3.17 Theorem: *(Continuous Functions are Integrable) Let $f:[a,b]\to\mathbb{R}$ be continuous. Then f is integrable on $[a,b]$.*

Proof: Let $\epsilon>0$. Since f is uniformly continuous on $[a,b]$, we can choose $\delta>0$ such that for all $x,y\in[a,b]$ we have $|x-y|<\delta \implies |f(x)-f(y)|<\frac{\epsilon}{b-a}$. Let X be any partition of $[a,b]$ with $|X|<\delta$. By the Extreme Value Theorem we have $M_i=f(t_i)$ and $m_i=f(s_i)$ for some $t_i,s_i\in[x_{i-1},x_i]$. Since $|t_i-s_i|\le|x_i-x_{i-1}|\le|X|=\delta$, we have $|M_i-m_i|=|f(t_i)-f(s_i)|<\frac{\epsilon}{b-a}$. Thus

$$U(f,X)-L(f,X)=\sum_{i=1}^n M_i\Delta_i x-\sum_{i=1}^n m_i\Delta_i x=\sum_{i=1}^n (M_i-m_i)\Delta_i x<\frac{\epsilon}{b-a}\sum_{i=1}^n\Delta_i x=\epsilon\,.$$

3.18 Note: Let f be integrable on $[a,b]$. Let X_n be any sequence of partitions of $[a,b]$ with $\lim\limits_{n\to\infty}|X_n|=0$. Let S_n be any Riemann sum for f on X_n. Then $\{S_n\}$ converges with

$$\lim_{n\to\infty}S_n=\int_a^b f(x)\,dx\,.$$

Proof: Write $I=\int_a^b f$. Given $\epsilon>0$, choose $\delta>0$ so that for every partition X of $[a,b]$ with $|X|<\delta$ we have $|S-I|<\epsilon$ for every Riemann sum S for f on X, and then choose N so that $n>N\implies|X_n|<\delta$. Then we have $n>N\implies|S_n-I|<\epsilon$.

3.19 Note: Let f be integrable on $[a,b]$. If we let X_n be the partition of $[a,b]$ into n equal-sized subintervals, and we let S_n be the Riemann sum on X_n using right-endpoints, then by the above note we obtain the formula

$$\int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{i=1}^n f(x_{n,i})\Delta_{n,i}x\ ,\text{ where } x_{n,i}=a+\tfrac{b-a}{n}\,i \text{ and } \Delta_{n,i}x=\tfrac{b-a}{n}\,.$$

3.20 Example: Find $\displaystyle\int_0^2 2^x\,dx$.

Solution: Let $f(x)=2^x$. Note that f is continuous and hence integrable, so we have

$$\begin{aligned}\int_0^2 2^x\,dx&=\lim_{n\to\infty}\sum_{i=1}^n f(x_{n,i})\Delta_{n,i}x=\lim_{n\to\infty}\sum_{i=1}^n f\left(\tfrac{2i}{n}\right)\left(\tfrac{2}{n}\right)=\lim_{n\to\infty}\sum_{i=1}^n 2^{2i/n}\left(\tfrac{2}{n}\right)\\&=\lim_{n\to\infty}\frac{2\cdot4^{1/n}}{n}\cdot\frac{4-1}{4^{1/n}-1}\ ,\text{ by the formula for the sum of a geometric sequence}\\&=\Big(\lim_{n\to\infty}6\cdot4^{1/n}\Big)\Big(\lim_{n\to\infty}\frac{1}{n\left(4^{1/n}-1\right)}\Big)=6\lim_{n\to\infty}\frac{\frac{1}{n}}{4^{1/n}-1}=6\lim_{x\to0}\frac{x}{4^x-1}\\&=6\lim_{x\to0}\frac{1}{\ln4\cdot4^x}\ ,\text{ by l'Hôpital's Rule}\\&=\tfrac{6}{\ln4}=\tfrac{3}{\ln2}\,.\end{aligned}$$

3.21 Lemma: *(Summation Formulas) We have*

$$\sum_{i=1}^{n} 1 = n \ , \ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \ , \ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \ , \ \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Proof: These formulas could be proven by induction, but we give a more constructive proof. It is obvious that $\sum\limits_{i=1}^{n} 1 = 1+1+\cdots 1 = n$. To find $\sum\limits_{i=1}^{n} i$, consider $\sum\limits_{n=1}^{n} \left(i^2-(i-1)^2\right)$. On the one hand, we have

$$\begin{aligned}\sum_{i=1}^{n} \left(i^2-(i-1)^2\right) &= (1^2-0^2)+(2^2-1^2)+\cdots+((n-1)^2-(n-2)^2)+(n^2-(n-1)^2) \\ &= -0^2+(1^2-1^2)+(2^2-2^2)+\cdots+((n-1)^2-(n-1)^2)+n^2 \\ &= n^2\end{aligned}$$

and on the other hand,

$$\sum_{i=1}^{n} \left(i^2-(i-1)^2\right) = \sum_{i=1}^{n} \left(i^2-(i^2-2i+1)\right) = \sum_{i=1}^{n}(2i-1) = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

Equating these gives $n^2 = 2\sum\limits_{i=1}^{n} i - \sum\limits_{i=1}^{n} 1$ and so

$$2\sum_{i=1}^{n} i = n^2 + \sum_{i=1}^{n} 1 = n^2+n = n(n+1)\,,$$

as required. Next, to find $\sum\limits_{n=1}^{\infty} i^2$, consider $\sum\limits_{i=1} \left(i^3-(i-1)^3\right)$. On the one hand we have

$$\begin{aligned}\sum_{i=1}^{n} \left(i^3-(i-1)^3\right) &= (1^3-0^3)+(2^3-1^3)+(3^3-2^3)+\cdots+(n^3-(n-1)^3) \\ &= -0^3+(1^3-1^3)+(2^3-2^3)+\cdots+((n-1)^3-(n-1)^3)+n^3 \\ &= n^3\end{aligned}$$

and on the other hand,

$$\begin{aligned}\sum_{i=1}^{n} \left(i^3-(i-1)^3\right) &= \sum_{i=1}^{n} \left(i^3-(i^3-3i^2+3i-1)\right) \\ &= \sum_{i=1}^{n}(3i^2-3i+1) = 3\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\,.\end{aligned}$$

Equating these gives $n^3 = 3\sum\limits_{i=1}^{n} i^2 - 3\sum\limits_{i=1}^{n} i + \sum\limits_{i=1}^{n} 1$ and so

$$6\sum_{i=1}^{n} i^2 = 2n^3 + 6\sum_{i=1}^{n} i - 2\sum_{i=1}^{n} 1 = 2n^3+3n(n+1)-2n = n(n+1)(2n+1)$$

as required. Finally, to find $\sum\limits_{i=1}^{n} i^3$, consider $\sum\limits_{i=1}^{n} \left(i^4-(i-1)^4\right)$. On the one hand we have

$$\sum_{i=1}^{n} \left(i^4-(i-1)^4\right) = n^4\,,$$

(as above) and on the other hand we have

$$\sum_{i=1}^{n} \left(i^4-(i-1)^4\right) = \sum_{i=1}^{n}(4i^3-6i^2+4i-1) = 4\sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\,.$$

Equating these gives $n^4 = 4\sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$ and so

$$\begin{aligned} 4\sum_{i=1}^{n} i^3 &= n^4 + 6\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ &= n^4 + n(n+1)(2n+1) - 2n(n+1) + n \\ &= n^4 + 2n^3 + n^2 = n^2(n+1)^2\,, \end{aligned}$$

as required.

3.22 Example: Find $\displaystyle\int_1^3 x + 2x^3\ dx$.

Solution: Let $f(x) = x + 2x^3$. Then

$$\begin{aligned} \int_1^3 x + 2x^3\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{n,i})\Delta_{n,i}x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \tfrac{2}{n}\, i\right)\left(\tfrac{2}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(1 + \tfrac{2}{n}\, i\right) + 2\left(1 + \tfrac{2}{n}\, i\right)^3\right)\left(\tfrac{2}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \tfrac{2}{n}\, i + 2\left(1 + \tfrac{6}{n}\, i + \tfrac{12}{n^2}\, i^2 + \tfrac{8}{n^3}\, i^3\right)\right)\left(\tfrac{2}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\tfrac{6}{n} + \tfrac{28}{n^2}\, i + \tfrac{48}{n^3}\, i^2 + \tfrac{32}{n^4}\, i^3\right) \\ &= \lim_{n\to\infty} \left(\tfrac{6}{n}\sum_{i=1}^{n} 1 + \tfrac{28}{n^2}\sum_{i=1}^{n} i + \tfrac{48}{n^3}\sum_{i=1}^{n} i^2 + \tfrac{32}{n^4}\sum_{i=1}^{n} i^3\right) \\ &= \lim_{n\to\infty} \left(\tfrac{6}{n}\cdot n + \tfrac{28}{n^2}\cdot\tfrac{n(n+1)}{2} + \tfrac{48}{n^3}\cdot\tfrac{n(n+1)(2n+1)}{6} + \tfrac{32}{n^4}\cdot\tfrac{n^2(n+1)^2}{4}\right) \\ &= 6 + \tfrac{28}{2} + \tfrac{48\cdot 2}{6} + \tfrac{32}{4} = 44\,. \end{aligned}$$

Basic Properties of Integrals

3.23 Theorem: *(Linearity) Let f and g be integrable on $[a,b]$ and let $c \in \mathbb{R}$. Then $f+g$ and cf are both integrable on $[a,b]$ and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

and

$$\int_a^b cf = c\int_a^b f\,.$$

Proof: The proof is left as an exercise.

3.24 Theorem: *(Comparison) Let f and g be integrable on $[a,b]$. If $f(x) \leq g(x)$ for all $x \in [a,b]$ then*

$$\int_a^b f \leq \int_a^b g\,.$$

Proof: The proof is left as an exercise.

3.25 Theorem: *(Additivity) Let $a<b<c$ and let $f:[a,c]\to\mathbb{R}$ be bounded. Then f is integrable on $[a,c]$ if and only if f is integrable both on $[a,b]$ and on $[b,c]$, and in this case*

$$\int_a^b f + \int_b^c f = \int_a^c f\,.$$

Proof: Suppose that f is integrable on $[a,c]$. Choose a partition X of $[a,c]$ such that $U(f,X)-L(f,X)<\epsilon$. Say that $b\in[x_{i-1},x_i]$ and let $Y=\{x_0,x_1,\cdots,x_{i-1},b\}$ and $Z=\{b,x_i,x_{i+1},\cdots,x_n\}$ so that Y and Z are partitions of $[a,b]$ and of $[b,c]$. Then we have $U(f,Y)-L(f,Y)\leq U(f,X\cup\{b\})-L(f,X\cup\{b\})\leq U(f,X)-L(f,X)<\epsilon$ and also $U(f,Z)-L(f,Z)\leq U(f,X\cup\{b\})-L(f,X\cup\{b\})\leq U(f,X)-L(f,X)<\epsilon$ and so f is integrable both on $[a,b]$ and on $[b,c]$.

Conversely, suppose that f is integrable both on $[a,b]$ and on $[b.c]$. Choose a partition Y of $[a,b]$ so that $U(f,Y)-L(f,Y)<\frac{\epsilon}{2}$ and choose a partition Z of $[b,c]$ such that $U(f,Z)-L(f,Z)<\frac{\epsilon}{2}$. Let $X=Y\cup Z$. Then X is a partition of $[a,c]$ and we have $U(f,X)-L(f,X)=\big(U(f,Y)+U(f,Z)\big)-\big(L(f,Y)+L(f,Z)\big)<\epsilon$.

Now suppose that f is integrable on $[a,c]$ (hence also on $[a,b]$ and on $[b,c]$) with $I_1=\displaystyle\int_a^b f$, $I_2=\displaystyle\int_b^c f$ and $I=\displaystyle\int_a^c f$. Let $\epsilon>0$. Choose $\delta>0$ so that for all partitions X_1, X_2 and X of $[a,b]$, $[b,c]$ and $[a,c]$ respectively with $|X_1|<\delta$, $|X_2|<\delta$ and $|X|<\delta$, we have $|S_1-I_1|<\frac{\epsilon}{3}$, $|S_2-I_2|<\frac{\epsilon}{3}$ and $|S-I|<\frac{\epsilon}{3}$ for all Riemann sums S_1, S_2 and S for f on X_1, X_2 and X respectively. Choose partitions X_1 and X_2 of $[a,b]$ and $[b,c]$ with $|X_1|<\delta$ and $|X_2|<\delta$. Choose Riemann sums S_1 and S_2 for f on X_1 and X_2. Let $X=X_1\cup X_2$ and note that $|X|<\delta$ and that $S=S_1+S_2$ is a Riemann sum for f on X. Then we have

$$\big|I-(I_1+I_2)\big| = \big|(I-S)+(S_1-I_1)+(S_2-I_2)\big| \leq \big|I-S\big|+|S_1-I_1|+|S_2-I_2| \leq \tfrac{\epsilon}{3}+\tfrac{\epsilon}{3}+\tfrac{\epsilon}{3}=\epsilon.$$

3.26 Definition: We define $\int_a^a f = 0$ and for $a < b$ we define $\int_b^a f = -\int_a^b f$.

3.27 Note: Using the above definition, the Additivity Theorem extends to the case that $a, b, c \in \mathbb{R}$ are not in increasing order: for any $a, b, c \in \mathbb{R}$, if f is integrable on $\big[\min\{a,b,c\}, \max\{a,b,c\}\big]$ then

$$\int_a^b f + \int_b^c f = \int_a^c f\,.$$

3.28 Theorem: *(Integration and Absolute Value) Let f be integrable on $[a,b]$. Then $|f|$ is integrable on $[a,b]$ and*

$$\left|\int_a^b f\right| \le \int_a^b |f|\,.$$

Proof: Let $\epsilon > 0$. Choose a partition X of $[a,b]$ such that $U(f,X) - L(f,X) < \epsilon$. Write $M_i(f) = \sup\big\{f(t)\big|t \in [x_{i-1}, x_i]\big\}$ and $M_i(|f|) = \sup\big\{|f(t)|\big|t \in [x_{i-1}, x_i]\big\}$, and similarly for $m_i(f)$ and $m_i(|f|)$.

When $0 \le m_i(f) \le M_i(f)$ we have $M_i(|f|) = M_i(f)$ and $m_i(|f|) = m_i(f)$. When $m_i(f) \le 0 \le M_i(f)$ we have $M_i(|f|) = \max\{M_i(f), -m_i(f)\}$ and $m_i(|f|) \ge 0$, and so $M_i(|f|) - m_i(|f|) \le \max\big\{M_i(f), -m_i(f)\big\} \le M_i(f) - m_i(f)$. When $m_i(f) \le M_i(f) \le 0$ we have $M_i(|f|) = -m_i(f)$ and $m_i(|f|) = -M_i(f)$, and so $M_i(|f|) - m_i(|f|) = M_i(f) - m_i(f)$. In all three cases we have

$$M_i(|f|) - m_i(|f|) \le M_i(f) - m_i(f)$$

and so

$$\begin{aligned} U(|f|, X) - L(|f|, X) &= \sum_{i=1}^n \big(M_i(|f|) - m_i(|f|)\big)\Delta_i x \le \sum_{i=1}^n \big(M_i(f) - m_i(f)\big)\Delta_i x \\ &= U(f,X) - L(f,X) < \epsilon\,. \end{aligned}$$

Thus $|f|$ is integrable on $[a,b]$.

Again, let $\epsilon > 0$. Choose a partition X on $[a,b]$ and choose values $t_i \in [x_{i-1}, x_i]$ so that

$$\left|\sum_{i=1}^n f(t_i)\Delta_i x - \int_a^b f\right| < \tfrac{\epsilon}{2} \text{ and } \left|\sum_{i=1}^n |f(t_i)|\Delta_i x - \int_a^b |f|\right| < \tfrac{\epsilon}{2}\,.$$

Note that by the triangle inequality we have $\left|\sum\limits_{i=1}^n f(t_i)\Delta_i x\right| \le \sum\limits_{i=1}^n |f(t_i)|\Delta_i x$, and so

$$\begin{aligned} \left|\int_a^b f\right| - \int_a^b |f| &= \left(\left|\int_a^b f\right| - \left|\sum_{i=1}^n f(t_i)\Delta_i x\right|\right) + \left(\left|\sum_{i=1}^n f(t_i)\Delta_i x\right| - \sum_{i=1}^n |f(t_i)|\Delta_i x\right) \\ &\quad + \left(\sum_{i=1}^n |f(t_i)|\Delta_i x - \int_a^b |f|\right) \\ &< \tfrac{\epsilon}{2} + 0 + \tfrac{\epsilon}{2} = \epsilon \end{aligned}$$

Since $\left|\int_a^b f\right| - \int_a^b |f| < \epsilon$ for every $\epsilon > 0$, we have $\left|\int_a^b f\right| - \int_a^b |f| \le 0$, as required.

The Fundamental Theorem of Calculus

3.29 Notation: For a function F, defined on an interval containing $[a,b]$, we write

$$\Big[F(x)\Big]_a^b = F(b) - F(a)\,.$$

3.30 Theorem: *(The Fundamental Theorem of Calculus)*
(1) Let f be integrable on $[a,b]$. Define $F:[a,b]\to\mathbb{R}$ by

$$F(x) = \int_a^x f = \int_a^x f(t)\,dt\,.$$

Then F is continuous on $[a,b]$. Moreover, if f is continuous at a point $x\in[a,b]$ then F is differentiable at x and

$$F'(x) = f(x)\,.$$

(2) Let f be integrable on $[a,b]$. Let F be differentiable on $[a,b]$ with $F'=f$. Then

$$\int_a^b f = \Big[F(x)\Big]_a^b = F(b) - F(a)\,.$$

Proof: (1) Let M be an upper bound for $|f|$ on $[a,b]$. For $a\le x,y\le b$ we have

$$\big|F(y)-F(x)\big| = \left|\int_a^y f - \int_a^x f\right| = \left|\int_x^y f\right| \le \left|\int_x^y |f|\right| \le \left|\int_x^y M\right| = M|y-x|$$

so given $\epsilon>0$ we can choose $\delta=\frac{\epsilon}{M}$ to get

$$|y-x|<\delta \implies \big|F(y)-F(x)\big| \le M|y-x| < M\delta = \epsilon\,.$$

Thus F is continuous (indeed uniformly continuous) on $[a,b]$. Now suppose that f is continuous at the point $x\in[a,b]$. Note that for $a\le x,y\le b$ with $x\ne y$ we have

$$\begin{aligned}
\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| &= \left|\frac{\int_a^y f - \int_a^x f}{y-x} - f(x)\right| \\
&= \left|\frac{\int_x^y f}{y-x} - \frac{\int_x^y f(x)}{y-x}\right| \\
&= \frac{1}{|y-x|}\left|\int_x^y \big(f(t)-f(x)\big)\,dt\right| \\
&\le \frac{1}{|y-x|}\left|\int_x^y \big|f(t)-f(x)\big|\,dt\right|\,.
\end{aligned}$$

Given $\epsilon>0$, since f is continuous at x we can choose $\delta>0$ so that

$$|y-x|<\delta \implies \big|f(y)-f(x)\big|<\epsilon$$

and then for $0<|y-x|<\delta$ we have

$$\begin{aligned}
\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| &\le \frac{1}{|y-x|}\left|\int_x^y \big|f(t)-f(x)\big|\,dt\right| \\
&\le \frac{1}{|y-x|}\left|\int_x^y \epsilon\,dt\right| = \frac{1}{|y-x|}\,\epsilon|y-x| = \epsilon\,.
\end{aligned}$$

and thus we have $F'(x)=f(x)$ as required.

(2) Let f be integrable on $[a,b]$. Suppose that F is differentiable on $[a,b]$ with $F'=f$. Let $\epsilon>0$ be arbitrary. Choose $\delta>0$ so that for every partition X of $[a,b]$ with $|X|<\delta$ we have $\left|\int_a^b f-\sum_{i=1}^n f(t_i)\Delta_i x\right|<\epsilon$ for every choice of sample points $t_i\in[x_{i-1},x_i]$. Choose sample points $t_i\in[x_{i-1},x_i]$ as in the Mean Value Theorem so that

$$F'(t_i)=\frac{F(x_i)-F(x_{i-1})}{x_i-x_{i-1}}\,,$$

that is $f(t_i)\Delta_i x=F(x_i)-F(x_{i-1})$. Then $\left|\int_a^b f-\sum_{i=1}^n f(t_i)\Delta_i x\right|<\epsilon$, and

$$\begin{aligned}\sum_{i=1}^n f(t_i)\Delta_i x&=\sum_{i=1}^n\big(F(x_i)-F(x_{i-1})\big)\\&=\big(F(x_1)-F(x)\big)+\big(F(x_2)-F(x_1)\big)+\cdots+\big(F(n-1)-F(x_n)\big)\\&=-F(x)+\big(F(x_1)-F(x_1)\big)+\cdots+\big(F(x_{n-1})-F(x_{n-1})\big)+F(x_n)\\&=F(x_n)-F(x)=F(b)-F(a)\,.\end{aligned}$$

and so $\left|\int_a^b f-\big(F(b)-F(a)\big)\right|<\epsilon$. Since ϵ was arbitrary, $\left|\int_a^b f-\big(F(b)-F(a)\big)\right|=0$.

3.31 Definition: A function F such that $F'=f$ on an interval is called an **antiderivative** of f on the interval.

3.32 Note: If $G'=F'=f$ on an interval, then $(G-F)'=0$, and so $G-F$ is constant on the interval, that is $G=F+c$ for some constant c.

3.33 Notation: We write

$$\int f=F+c\ ,\ \text{or}\ \int f(x)\,dx=F(x)+c$$

when F is an antiderivative of f on an interval, so that the antiderivatives of f on the interval are the functions of the form $G=F+c$ for some constant c.

3.34 Example: Find $\displaystyle\int_0^{\sqrt{3}}\frac{dx}{1+x^2}$.

Solution: We have $\displaystyle\int\frac{dx}{1+x^2}=\tan^{-1}x+c$, since $\displaystyle\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$, and so by Part 2 of the Fundamental Theorem of Calculus, we have

$$\int_0^{\sqrt{3}}\frac{dx}{1+x^2}=\Big[\tan^{-1}x\Big]_0^{\sqrt{3}}=\tan^{-1}\sqrt{3}-\tan^{-1}0=\tfrac{\pi}{3}\,.$$