Chapter 4. Sequences and Series of Functions

Pointwise Convergence

4.1 Definition: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and for each integer $n \geq p$ let $f_n : A \to \mathbb{R}$. We say that the sequence of functions $(f_n)_{n \geq p}$ **converges pointwise** to f on A, and we write $f_n \to f$ pointwise on A, when $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in A$, that is when for all $x \in A$ and for all $\epsilon > 0$ there exists $m \geq p$ such that for all integers n we have

$$n \geq m \Longrightarrow |f_n(x) - f(x)| < \epsilon\,.$$

4.2 Note: By the Cauchy Criterion for convergence, the sequence $(f_n)_{n\geq p}$ converges pointwise to some function $f(x)$ on A if and only if for all $x \in A$ and for all $\epsilon > 0$ there exists $m \geq p$ such that for all integers k, ℓ we have

$$k, \ell \geq m \Longrightarrow \big|f_k(x) - f_\ell(x)\big| < \epsilon\,.$$

4.3 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is continuous but f is not.

Solution: Let $f_n(x) = x^n$. Then $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 \text{ if } x \neq 1 \\ 1 \text{ if } x = 1\,. \end{cases}$

4.4 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is differentiable and f is differentiable, but $\lim_{n\to\infty} {f_n}' \neq f'$.

Solution: Let $f_n(x) = \frac{1}{n}\tan^{-1}(nx)$. Then $\lim_{n\to\infty} f_n(x) = 0$, and ${f_n}'(x) = \dfrac{1}{1+(nx)^2}$ so $\lim_{n\to\infty} {f_n}'(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ 1 \text{ if } x = 0\,. \end{cases}$

4.5 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is integrable but f is not.

Solution: We have $\mathbb{Q} \cap [0,1] = \{a_1, a_2, a_3, \cdots\}$ where

$$(a_n)_{n\geq 1} = \left(\tfrac{0}{1}, \tfrac{1}{1}, \tfrac{0}{2}, \tfrac{1}{2}, \tfrac{2}{2}, \tfrac{0}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, \tfrac{0}{4}, \cdots, \tfrac{4}{4}, \cdots\right).$$

(as an exercise, you can check that $a_n = \frac{k}{\ell}$ where $\ell = \left\lceil \frac{-3+\sqrt{9-8n}}{2} \right\rceil$ and $k = n - \frac{\ell^2+\ell}{2}$). For $x \in [0,1]$, let $f_n(x) = \begin{cases} 0 \text{ if } x \notin \{a_1, a_2, \cdots, a_n\} \\ 1 \text{ if } x \in \{a_1, a_2, \cdots, a_n\}\,. \end{cases}$ Then $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 \text{ if } x \notin \mathbb{Q} \\ 1 \text{ if } x \in \mathbb{Q}\,. \end{cases}$

4.6 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is integrable and f is integrable but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx\,.$$

Solution: Let $f_1(x) = \begin{cases} 48\left(x - \frac{1}{2}\right)(1-x) \text{ if } \frac{1}{2} \leq x \leq 1, \\ 0 \text{ otherwise}\,. \end{cases}$ For $n \geq 1$ let $f_n(x) = n f_1(nx)$.

Then each f_n is continuous with $\int_0^1 f_n(x)\,dx = 1$, and $\lim_{n\to\infty} f_n(x) = 0$ for all x.

Uniform Convergence

4.7 Definition: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and for each integer $n \geq p$ let $f_n : A \to \mathbb{R}$. We say that the sequence of functions $(f_n)_{n \geq p}$ **converges uniformly** to f on A, and we write $f_n \to f$ uniformly on A, when for all $\epsilon > 0$ there exists $m \in \mathbb{Z}_{\geq p}$ such that for all $x \in A$ and for all integers $n \in \mathbb{Z}_{\geq p}$ we have

$$n \geq m \Longrightarrow \left|f_n(x) - f(x)\right| < \epsilon\,.$$

4.8 Theorem: *(Cauchy Criterion for Uniform Convergence of Sequences of Functions) Let $(f_n)_{n \geq p}$ be a sequence of functions on $A \subseteq \mathbb{R}$. Then (f_n) converges uniformly (to some function f) on A if and only if for all $\epsilon > 0$ there exists $m \in \mathbb{Z}_{\geq p}$ such that for all $x \in A$ and for all integers $k, \ell \in \mathbb{Z}_{\geq p}$ we have*

$$k, \ell \geq m \Longrightarrow \left|f_k(x) - f_\ell(x)\right| < \epsilon\,.$$

Proof: Suppose that (f_n) converges uniformly to f on A. Let $\epsilon > 0$. Choose m so that for all $x \in A$ we have $n \geq m \Longrightarrow \left|f_n(x) - f(x)\right| < \frac{\epsilon}{2}$. Then for $k, \ell \geq m$ we have $\left|f_k(x) - f(x)\right| < \frac{\epsilon}{2}$ and $\left|f_\ell(x) - f(x)\right| < \frac{\epsilon}{2}$ and so

$$\left|f_k(x) - f_\ell(x)\right| \leq \left|f_k(x) - f(x)\right| + \left|f_\ell(x) - f(x)\right| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,.$$

Conversely, suppose that (f_n) satisfies the Cauchy Criterion for uniform convergence, that is for all $\epsilon > 0$ there exists m such that for all $x \in A$ and all integers n, ℓ we have

$$n, \ell \geq m \Longrightarrow \left|f_n(x) - f_\ell(x)\right| < \epsilon\,.$$

For each fixed $x \in A$, $(f_n(x))$ is a Cauchy sequence, so $(f_n(x))$ converges, and we can define $f(x)$ by

$$f(x) = \lim_{n \to \infty} f_n(x)\,.$$

We know that $f_n \to f$ pointwise on A, but we must show that $f_n \to f$ uniformly on A. Let $\epsilon > 0$. Choose m so that for all $x \in A$ and for all integers n, ℓ we have

$$n, \ell \geq m \Longrightarrow \left|f_n(x) - f_\ell(x)\right| < \tfrac{\epsilon}{2}\,.$$

Let $x \in A$. Since $\lim\limits_{\ell \to \infty} f_\ell(x) = f(x)$, we can choose $\ell \geq m$ so that $\left|f_\ell(x) - f(x)\right| < \frac{\epsilon}{2}$. Then for $n \geq m$ we have

$$\left|f_n(x) - f(x)\right| \leq \left|f_n(x) - f_\ell(x)\right| + \left|f_\ell(x) - f(x)\right| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,.$$

4.9 Theorem: *(Uniform Convergence, Limits and Continuity) Suppose that $f_n \to f$ uniformly on A. Let x be a limit point of A. If $\lim\limits_{y \to x} f_n(y)$ exists for each n, then*

$$\lim_{y \to x} \lim_{n \to \infty} f_n(y) = \lim_{n \to \infty} \lim_{y \to x} f_n(y)\,.$$

In particular, if each f_n is continuous in A, then so is f.

Proof: Suppose that $\lim\limits_{y \to x} f_n(y)$ exists for all n. Let $b_n = \lim\limits_{y \to x} f_n(y)$. We must show that $\lim\limits_{y \to x} f(y) = \lim\limits_{n \to \infty} b_n$. We claim first that (b_n) converges. Let $\epsilon > 0$. Choose m so that $k, \ell \geq m \Longrightarrow \left|f_k(y) - f_\ell(y)\right| < \frac{\epsilon}{3}$ for all $y \in A$. Let $k, \ell \geq m$. Choose $y \in A$ so that $\left|f_k(y) - b_k\right| < \frac{\epsilon}{3}$ and $\left|f_\ell(y) - b_\ell\right| < \frac{\epsilon}{3}$. Then we have

$$\left|b_k - b_\ell\right| \leq \left|b_k - f_k(y)\right| + \left|f_k(y) - f_\ell(y)\right| + \left|f_\ell(y) - b_\ell\right| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon\,.$$

By the Cauchy Criterion for sequences, (b_n) converges, as claimed.

Now, let $b = \lim_{n\to\infty} b_n$. We must show that $\lim_{y\to x} f(x) = b$. Let $\epsilon > 0$. Choose m so that when $n \geq m$ we have $\big|f_n(y) - f(y)\big| < \frac{\epsilon}{3}$ for all $y \in A$ and we have $|b_n - b| < \frac{\epsilon}{3}$. Let $n \geq m$. Since $\lim_{y\to x} f_n(y) = b_n$ we can choose $\delta > 0$ so that $0 < |y - x| < \delta \Longrightarrow \big|f_n(y) - b_n\big| < \frac{\epsilon}{3}$. Then when $0 < |y - x| < \delta$ we have

$$\big|f(y) - b\big| \leq \big|f(y) - f_n(y)\big| + \big|f_n(y) - b_n\big| + \big|b_n - b\big| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon\,.$$

Thus $\lim_{y\to x} f(x) = b$, as required.

In particular, if $x \in A$ and each f_n is continuous at x then we have

$$\lim_{y\to x} f(y) = \lim_{y\to x}\lim_{n\to\infty} f_n(y) = \lim_{n\to\infty}\lim_{y\to x} f_n(y) = \lim_{n\to\infty} f_n(x) = f(x)$$

so f is continuous at x.

4.10 Theorem: *(Uniform Convergence and Integration) Suppose that $f_n \to f$ uniformly on $[a,b]$. If each f_n is integrable on $[a,b]$ then so is f. In this case, if $g_n(x) = \int_a^x f_n(t)\,dt$ and $g(x) = \int_a^x f(t)\,dt$, then $g_n \to g$ uniformly on $[a,b]$. In particular, we have*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx\,.$$

Proof: Suppose that each f_n is integrable on $[a,b]$. We claim that f is integrable on $[a,b]$. Let $\epsilon > 0$. Choose N so that $n \geq N \Longrightarrow \big|f_n(x) - f(x)\big| < \frac{\epsilon}{4(b-a)}$ for all $x \in [a,b]$. Fix $n \geq N$. Choose a partition X of $[a,b]$ so that $U(f_n, X) - L(f_n, X) < \frac{\epsilon}{2}$. Note that since $\big|f_n(x) - f(x)\big| < \frac{\epsilon}{4(b-a)}$ we have $M_i(f) < M_i(f_n) + \frac{\epsilon}{4(b-a)}$ and $m_i(f) > m_i(f_n) - \frac{\epsilon}{4(b-a)}$, and so

$$\begin{aligned} U(f,X) - L(f,X) &= \sum_{i=1}^{n} \big(M_i(f) - m_i(f)\big)\Delta_i x < \sum_{i=1}^{n} \Big(M_i(f_n) - m_i(f_n) + \tfrac{\epsilon}{2(b-a)}\Big)\,\Delta_i x \\ &= U(f_n, X) - L(f_n, X) + \tfrac{\epsilon}{2} < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,. \end{aligned}$$

Thus f is integrable on $[a,b]$.

Now define $g_n(x) = \int_a^x f_n(t)\,dt$ and $g(x) = \int_a^x f(t)\,dt$. We claim that $g_n \to g$ uniformly on $[a,b]$. Let $\epsilon > 0$. Choose N so that $n \geq N \Longrightarrow \big|f_n(t) - f(t)\big| < \frac{\epsilon}{2(b-a)}$ for all $t \in I$. Let $n \geq N$. Let $x \in [a,b]$. Then we have

$$\begin{aligned} \big|g_n(x) - g(x)\big| &= \left|\int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt\right| = \left|\int_a^x f_n(t) - f(t)\ dt\right| \\ &\leq \int_a^x \big|f_n(t) - f(t)\big|\,dt \leq \int_a^x \tfrac{\epsilon}{2(b-a)}\ dt = \tfrac{\epsilon}{2(b-a)}(x-a) \leq \tfrac{\epsilon}{2} < \epsilon\,. \end{aligned}$$

Thus $g_n \to g$ uniformly on $[a,b]$, as required.

In particular, we have $\lim_{n\to\infty} g_n(b) = g(b)$, that is

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx\,.$$

4.11 Theorem: *(Uniform Convergence and Differentiation) Let (f_n) be a sequence of functions on $[a,b]$. Suppose that each f_n is differentiable on $[a,b]$, $\big(f_n{}'\big)$ converges uniformly on $[a,b]$, and $\big(f_n(c)\big)$ converges for some $c\in[a,b]$. Then (f_n) converges uniformly on $[a,b]$, $\lim\limits_{n\to\infty} f_n(x)$ is differentiable, and*

$$\frac{d}{dx}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{d}{dx} f_n(x)\,.$$

Proof: We claim that (f_n) converges uniformly on $[a,b]$. Let $\epsilon>0$. Choose N so that when $n,m\ge N$ we have $\big|f_n{}'(t)-f_m{}'(t)\big|<\frac{\epsilon}{2(b-a)}$ for all $t\in[a,b]$ and we have $\big|f_n(c)-f_m(c)\big|<\frac{\epsilon}{2}$. Let $n,m\ge N$. Let $x\in[a,b]$. By the Mean Value Theorem applied to the function $f_n(x)-f_m(x)$, we can choose t between c and x so that

$$\big(f_n(x)-f_m(x)-f_n(c)+f_m(c)\big)=\big(f_n{}'(t)-f_m{}'(t)\big)(x-c)\,.$$

Then we have

$$\begin{aligned}\big|f_n(x)-f_m(x)\big| &\le \big|f_n(x)-f_m(x)-f_n(c)+f_m(c)\big|+\big|f_n(c)-f_m(c)\big|\\ &= \big|f_n{}'(t)-f_m{}'(t)\big||x-c|+\big|f_n(c)-f_m(c)\big|\\ &< \tfrac{\epsilon}{2(b-a)}(b-a)+\tfrac{\epsilon}{2}=\epsilon\,.\end{aligned}$$

Thus (f_n) converges uniformly on $[a,b]$.

Let $f(x)=\lim\limits_{n\to\infty} f_n(x)$. We claim that f is differentiable with $f'(x)=\lim\limits_{n\to\infty} f_n{}'(x)$ for all $x\in[a,b]$. Fix $x\in[a,b]$. Note that

$$\begin{aligned}f'(x)=\lim_{n\to\infty} f_n{}'(x) &\iff \lim_{y\to x}\frac{f(y)-f(x)}{y-x}=\lim_{n\to\infty}\lim_{y\to x}\frac{f_n(y)-f_n(x)}{y-x}\\ &\iff \lim_{y\to x}\lim_{n\to\infty}\frac{f_n(y)-f_n(x)}{y-x}=\lim_{n\to\infty}\lim_{y\to x}\frac{f_n(y)-f_n(x)}{y-x}\end{aligned}$$

so it suffices to show that $\big(g_n\big)$ converges uniformly on $[a,b]\setminus\{x\}$, where

$$g_n(y)=\frac{f_n(y)-f_n(x)}{y-x}\,.$$

Let $\epsilon>0$. Choose N so that $n,m\ge N\implies \big|f_n{}'(t)-f_m{}'(t)\big|<\epsilon$ for all $t\in[a,b]$. Let $n,m\ge N$. Let $y\in[a,b]\setminus\{x\}$. By the Mean Value Theorem, we can choose t between x and y so that

$$\big(f_n(y)-f_m(y)-f_n(x)+f_m(x)\big)=\big(f_n{}'(t)-f_m{}'(t)\big)(y-x)\,.$$

Then

$$\big|g_n(y)-g_m(y)\big|=\left|\frac{f_n(y)-f_m(y)-f_n(x)+f_m(x)}{y-x}\right|=\big|f_n{}'(t)-f_m{}'(t)\big|<\epsilon\,.$$

Thus $\big(g_n\big)$ converges uniformly on $[a,b]\setminus\{x\}$, as required.

Series of Functions

4.12 Definition: Let $(f_n)_{n\geq p}$ be a sequence of functions on $A \subseteq \mathbb{R}$. The **series of functions** $\sum_{n\geq p} f_n(x)$ is defined to be the sequence $\big(S_l(x)\big)$ where $S_l(x) = \sum_{n=p}^{l} f_n(x)$. The function $S_l(x)$ is called the l^{th} **partial sum** of the series. We say the series $\sum_{n\geq p} f_n(x)$ converges pointwise (or uniformly) on A when the sequence $\{S_l\}$ converges, pointwise (or uniformly) on A. In this case, the **sum** of the series of functions is defined to be the function

$$f(x) = \sum_{n=p}^{\infty} f_n(x) = \lim_{l\to\infty} S_l(x)\,.$$

4.13 Theorem: *(Cauchy Criterion for the Uniform Convergence of a Series of Functions) The series $\sum_{n\geq p} f_n(x)$ converges uniformly (to some function f) on A if and only if for every $\epsilon > 0$ there exists $N \geq p$ such that for all $x \in A$ and for all $k, \ell \geq p$ we have*

$$\ell > k \geq N \Longrightarrow \left| \sum_{n=k+1}^{\ell} f_n(x) \right| < \epsilon\,.$$

Proof: This follows immediately from the analogous theorem for sequences of functions.

4.14 Theorem: *(Uniform Convergence, Limits and Continuity) Suppose that $\sum_{n\geq p} f_n(x)$ converges uniformly on A. Let x be a limit point of A. If $\lim_{y\to x} f_n(y)$ exists for all $n \geq p$, then*

$$\lim_{y\to x} \sum_{n=p}^{\infty} f_n(y) = \sum_{n=p}^{\infty} \lim_{y\to x} f_n(y)\,.$$

In particular, if each $f_n(x)$ is continuous on A then so is $\sum_{n=p}^{\infty} f_n(x)$.

Proof: This follows immediately from the analogous theorem for sequences of functions.

4.15 Theorem: *(Uniform Convergence and Integration) Suppose that $\sum_{n\geq p} f_n(x)$ converges uniformly on $[a,b]$. If each $f_n(x)$ is integrable on $[a,b]$, then so is $\sum_{n=p}^{\infty} f_n(x)$. In this case, if we define $g_n(x) = \int_a^x f_n(t)\,dt$ and $g(x) = \int_a^x \sum_{n=p}^{\infty} f_n(t)\,dt$, then $\sum_{n\geq p} g_n(x)$ converges uniformly to $g(x)$ on A. In particular, we have*

$$\int_a^b \sum_{n=p}^{\infty} f_n(x)\,dx = \sum_{n=p}^{\infty} \int_a^b f_n(x)\,dx\,.$$

Proof: This follows immediately from the analogous theorem for sequences of functions.

4.16 Theorem: *(Uniform Convergence and Differentiation) Suppose that each $f_n(x)$ is differentiable on $[a,b]$, $\sum\limits_{n\geq p} {f_n}'(x)$ converges uniformly on $[a,b]$, and $\sum\limits_{n\geq p} f_n(c)$ converges for some $c\in[a,b]$. Then $\sum\limits_{n\geq p} f_n(x)$ converges uniformly on $[a,b]$ and*

$$\frac{d}{dx}\sum_{n=p}^{\infty} f_n(x) = \sum_{n=p}^{\infty}\frac{d}{dx} f_n(x)\,.$$

Proof: This follows immediately from the analogous theorem for sequences of functions.

4.17 Theorem: *(The Weierstrass M-Test) Suppose that f_n is bounded with $|f_n(x)|\leq M_n$ for all $n\geq p$ and $x\in A$, and $\sum\limits_{n\geq p} M_n$ converges. Then $\sum\limits_{n\geq p} f_n(x)$ converges uniformly on A.*

Proof: Let $\epsilon>0$. Choose N so that $\ell>k\geq N \implies \sum\limits_{n=k+1}^{\ell} M_n<\epsilon$. Let $\ell>k\geq N$. Let $x\in A$. Then

$$\left|\sum_{n=k+1}^{\ell} f_n(x)\right| \leq \sum_{n=k+1}^{\ell}\big|f_n(x)\big| \leq \sum_{n=k+1}^{\ell} M_n<\epsilon\,.$$

4.18 Example: Find a sequence of functions $\big(f_n(x)\big)_{n\geq 0}$, each of which is differentiable on $\mathbb{R}$, such that $\sum\limits_{n\geq 0} f_n(x)$ converges uniformly on $\mathbb{R}$, but the sum $f(x)=\sum\limits_{n=0}^{\infty} f_n(x)$ is nowhere differentiable.

Solution: Let $f_n(x)=\frac{1}{2^n}\sin^2(8^n x)$. Since $|f_n(x)|\leq\frac{1}{2^n}$ and $\sum\frac{1}{2^n}$ converges, $\sum\limits_{n\geq 0} f_n(x)$ converges uniformly on $\mathbb{R}$. Let $f(x)=\sum\limits_{n=0}^{\infty} f_n(x)$. We claim that $f(x)$ is nowhere differentiable. Let $x\in\mathbb{R}$. For each n, let m, a_n and b_n be such that $a_n=\frac{m\pi}{2\cdot 8^n}$, $b_n=\frac{(m+1)\pi}{2\cdot 8^n}$ and $x\in[a_n,b_n)$. Note that one of $f_n(a_n)$ and $f_n(b_n)$ is equal to $\frac{1}{2^n}$ and the other is equal to 0 so we have $\big|f_n(b_n)-f_n(a_n)\big|=\frac{1}{2^n}$. Note also that for $k>n$ we have $f_k(a_n)=f_k(b_n)=0$. Also, for all k we have $f_k(x)=\frac{1}{2^k}\sin^2(8^k x)$, ${f_k}'(x)=4^k\sin(2\cdot 8^k x)$, and $\big|{f_k}'(x)\big|\leq 4^k$, so by the Mean Value Theorem,

$$\big|f_k(b_n)-f_k(a_n)\big|\leq 4^k|b_n-a_n|\,.$$

Finally, note that if $f'(x)$ did exist, then we would have $f'(x)=\lim\limits_{n\to\infty}\dfrac{f(b_n)-f(a_n)}{b_n-a_n}$, but

$$\begin{aligned}\left|\frac{f(b_n)-f(a_n)}{b_n-a_n}\right| &= \left|\sum_{k=0}^{\infty}\frac{f_k(b_n)-f_k(a_n)}{b_n-a_n}\right| = \left|\sum_{k=0}^{n}\frac{f_k(b_n)-f_k(a_n)}{b_n-a_n}\right|\\ &\geq \left|\frac{f_n(b_n)-f_n(a_n)}{b_n-a_n}\right| - \sum_{k=0}^{n-1}\left|\frac{f_k(b_n)-f_k(a_n)}{b_n-a_n}\right|\\ &\geq \frac{\frac{1}{2^n}}{\frac{\pi}{2\cdot 8^n}} - \sum_{k=0}^{n-1}4^k = \tfrac{2\cdot 4^n}{\pi}-\tfrac{4^n-1}{3} = \left(\tfrac{2}{\pi}-\tfrac{1}{3}\right)4^n+\tfrac{1}{3}\to\infty \text{ as } n\to\infty\end{aligned}$$

Power Series

4.19 Definition: A **power series centred at** a is a series of the form $\sum_{n\geq 0} a_n(x-a)^n$ for some real numbers a_n, where we use the convention that $(x-a)^0 = 1$.

4.20 Example: The geometric series $\sum_{n\geq 0} x^n$ is a power series centred at 0. It converges when $|x|<1$ and for all such x the sum of the series is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,.$$

4.21 Lemma: *(Abel's Formula) Let $\{a_n\}$ and $\{b_n\}$ be sequences. Then we have*

$$\sum_{n=m}^{l} a_n b_n + \sum_{p=m}^{l-1}\left(\sum_{n=m}^{p} a_n\right)(b_{p+1}-b_p) = \left(\sum_{n=m}^{l} a_n\right) b_l\,.$$

Proof: We have

$$\begin{aligned}
\sum_{p=m}^{l-1}\left(\sum_{k=m}^{p} a_n\right)(b_{p+1}-b_p) =\ & a_m(b_{m+1}-b_m) + (a_m+a_{m+1})(b_{m+2}-b_{m+1}) \\
& + (a_m+a_{m+1}+a_{m+2})(b_{m+3}-b_{m+2}) \\
& + \cdots + (a_m+a_{m+1}+a_{m+2}+\cdots+a_{l-1})(b_l-b_{l-1}) \\
=\ & -a_m b_m - a_{m+1}b_{m+1} - \cdots - a_{l-1}b_{l-1} \\
& + (a_m+a_{m+1}+\cdots a_{l-1})b_l - a_l b_l + a_l b_l \\
=\ & \left(\sum_{n=m}^{l} a_n\right) b_l - \sum_{n=m}^{l} a_n b_n\,.
\end{aligned}$$

4.22 Remark: Let (a_n) be a sequence in $\mathbb{R}$. Recall that $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} s_n$ where $s_n = \sup\{a_k \,|\, k\geq n\}$ (with $\limsup_{n\to\infty} a_n = \infty$ when (a_n) is not bounded above).

4.23 Theorem: *(The Interval and Radius of Convergence) Let $\sum_{n\geq 0} a_n(x-a)^n$ be a power series and let $R = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \in [0,\infty]$. Then the set of $x\in\mathbb{R}$ for which the power series converges is an interval I centred at a of radius R. Indeed*

(1) if $|x-a|>R$ then $\lim_{n\to\infty} a_n(x-a)\neq 0$ so $\sum_{n\geq 0} n_n(x-a)^n$ diverges,

(2) if $|x-a|<R$ then $\sum_{n\geq 0} a_n(x-a)^n$ converges absolutely,

(3) if $0<r<R$ then $\sum_{n\geq 0} a_n(x-a)^n$ converges uniformly in $[a-r,a+r]$, and

(4) (Abel's Theorem) if $\sum_{n\geq 0} a_n(x-a)^n$ converges when $x=a+R$ then the convergence is uniform on $[a,a+R]$, and similarly if $\sum_{n\geq 0} a_n(x-a)^n$ converges when $x=a-R$ then the convergence is uniform on $[a-R,a]$.

Proof: To prove part (1), suppose that $|x-a|>R$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(x-a)^n|} = |x-a| \limsup_{n\to\infty} \sqrt[n]{|a_n|} > R\cdot \tfrac{1}{R} = 1\,,$$

and so $\lim\limits_{n\to\infty} a_n(x-a)^n \neq 0$ and $\sum a_n(x-a)^n$ diverges, by the Root Test.

To prove part (2), suppose that $|x-a|<R$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(x-a)^n|} = |x-a| \limsup_{n\to\infty} \sqrt[n]{|a_n|} < R\cdot \tfrac{1}{R} = 1\,,$$

and so $\sum \big|a_n(x-a)^n\big|$ converges, by the Root Test.

To prove part (3), fix $0<r<R$. By part (2), $\sum \big|a_n(x-a)^n\big|$ converges when $x=a+r$, that is $\sum |a_n r^n|$ converges. Let $x\in[a-r,a+r]$. Then $|a_n(x-a)^n|\le |a_n r^n|$ and $\sum |a_n r^n|$ converges, and so $\sum |a_n(x-a)^n|$ converges uniformly by the Weierstrass M-Test.

Now let us prove part (4). Suppose that $\sum a_n(x-a)^n$ converges when $x=a+R$, that is $\sum a_n R^n$ converges. Let $\epsilon>0$. Choose N so that $l>m>N \implies \left|\sum\limits_{n=m}^{l} a_n R^n\right| < \epsilon$. Then by Abel's Formula and using telescoping we have

$$\begin{aligned}
\left|\sum_{n=m}^{l} a_n(x-a)^n\right| &= \left|\sum_{n=m}^{l} a_n R^n \left(\tfrac{x-a}{R}\right)^n\right| \\
&= \left|\left(\sum_{n=m}^{l} a_n R^n\right)\left(\tfrac{x-a}{R}\right)^l - \sum_{p=m}^{l-1}\left(\sum_{n=m}^{p} a_n R^n\right)\left(\left(\tfrac{x-a}{R}\right)^{p+1} - \left(\tfrac{x-a}{R}\right)^p\right)\right| \\
&\le \left|\sum_{n=m}^{l} a_n R^n\right| \left(\tfrac{x-a}{R}\right)^l + \sum_{p=m}^{l-1}\left|\sum_{n=m}^{p} a_n R^n\right| \left(\left(\tfrac{x-a}{R}\right)^{p} - \left(\tfrac{x-a}{R}\right)^{p+1}\right) \\
&< \epsilon\left(\tfrac{x-a}{R}\right)^l + \epsilon\left(\left(\tfrac{x-a}{R}\right)^m - \left(\tfrac{x-a}{R}\right)^l\right) = \epsilon\left(\tfrac{x-a}{R}\right)^m < \epsilon\,.
\end{aligned}$$

4.24 Definition: The number R in the above theorem is called the **radius of convergence** of the power series, and the interval I is called the **interval of convergence** of the power series.

4.25 Example: Find the interval of convergence of the power series $\displaystyle\sum_{n\ge1} \frac{(3-2x)^n}{\sqrt{n}}$.

Solution: First note that this is in fact a power series, since $\dfrac{(3-2x)^n}{\sqrt{n}} = \frac{(-2)^n}{\sqrt{n}}\left(x-\frac{3}{2}\right)^n$, and so $\displaystyle\sum_{n\ge1}\frac{(3-2x)^n}{\sqrt{n}} = \sum_{n\ge0} c_n(x-a)^n$, where $c_0=0$, $c_n = \frac{(-2)^n}{\sqrt{n}}$ for $n\ge1$ and $a=\frac{3}{2}$.

Now, let $a_n = \dfrac{(3-2x)^n}{\sqrt{n}}$. Then $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(3-2x)^{n+1}}{\sqrt{n+1}}\,\dfrac{\sqrt{n}}{(3-2x)^n}\right| = \sqrt{\frac{n}{n+1}}\,|3-2x|$, so $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |3-2x|$. By the Ratio Test, $\sum a_n$ converges when $|3-2x|<1$ and diverges when $|3-2x|>1$. Equivalently, it converges when $x\in(1,2)$ and diverges when $x\notin[1,2]$. When $x=1$ so $(3-2x)=1$, we have $\sum a_n = \sum \frac{1}{\sqrt{n}}$, which diverges (its a p-series), and when $x=2$ so $(3-2x)=-1$, we have $\sum a_n = \sum \frac{(-1)^n}{\sqrt{n}}$ which converges by the Alternating Series Test. Thus the interval of convergence is $I=(1,2\,]$.

Operations on Power Series

4.26 Theorem: *(Continuity of Power Series) Suppose that the power series* $\sum a_n(x-a)^n$ *converges in an interval* I*. Then the sum* $f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$ *is continuous in* I*.*

Proof: This follows from uniform convergence of $\sum a_n(x-a)^n$ in closed subintervals of I.

4.27 Theorem: *(Addition and Subtraction of Power Series) Suppose that the power series* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in the interval* I*. Then* $\sum(a_n+b_n)(x-a)^n$ *and* $\sum(a_n-b_n)(x-a)^n$ *both converge in* I*, and for all* $x\in I$ *we have*

$$\left(\sum_{n=0}^{\infty} a_n(x-a)^n\right) \pm \left(\sum_{n=0}^{\infty} b_n(x-a)^n\right) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x-a)^n\,.$$

Proof: This follows from Linearity.

4.28 Theorem: *(Multiplication of Power Series) Suppose the power series* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in an open interval* I *with* $a\in I$*. Let* $c_n=\sum_{k=0}^{n} a_k b_{n-k}$*. Then* $\sum c_n(x-a)^n$ *converges in* I *and for all* $x\in I$ *we have*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \left(\sum_{n=0}^{\infty} a_n(x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-a)^n\right)\,.$$

Proof: This follows from the Multiplication of Series Theorem, since the power series converge absolutely in I.

4.29 Theorem: *(Division of Power Series) Suppose that* $\sum a_n(x-a)^n$ *and* $\sum b_n(x-a)^n$ *both converge in an open interval* I *with* $a\in I$*, and that* $b_0\neq 0$*. Define* c_n *by*

$$c_0=\tfrac{a_0}{b_0}\ \text{, and for } n>0,\ c_n=\tfrac{a_n}{b_0}-\tfrac{b_nc_0}{b_0}-\tfrac{b_{n-1}c_1}{b_0}-\cdots-\tfrac{b_1c_{n-1}}{b_0}\,.$$

Then there is an open interval J *with* $a\in J$ *such that* $\sum c_n(x-a)^n$ *converges in* J *and for all* $x\in J$*,*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \frac{\sum_{n=0}^{\infty} a_n(x-a)^n}{\sum_{n=0}^{\infty} b_n(x-a)^n}\,.$$

Proof: Choose $r>0$ so that $a+r\in I$. Note that $\sum|a_nr^n|$ and $\sum|b_nr^n|$ both converges. Since $|a_nr^n|\to 0$ and $|b_nr^n|\to 0$ and $b_0\neq 0$, we can choose M so that $M\geq\left|\frac{a_nr^n}{b_0}\right|$ and $M\geq\left|\frac{b_nr^n}{b_0}\right|$ for all n. Note that $|c_0|=\left|\frac{a_0}{b_0}\right|\leq M$ and since $c_1=\frac{a_1}{b_0}+\frac{b_1c_0}{b_0}$ we have

$$|c_1r|\leq\left|\tfrac{a_1r}{b_0}\right|+\left|\tfrac{b_1r}{b_0}\right||c_0|\leq M+M^2=M(1+M)\,.$$

Suppose, inductively, that $|c_kr^k|\leq M(1+M)^k$ for all $k<n$. Then since

$$a_n=b_nc_0+b_{n-1}c_1+\cdots+b_1c_{n-1}+b_0c_n\,,$$

we have

$$
\begin{aligned}
|c_n r^n| &\le \left|\tfrac{a_n r^n}{b_0}\right| + \left|\tfrac{b_n r^n}{b_0}\right| |c_0| + \left|\tfrac{b_{n-1} r^{n-1}}{b_0}\right| |c_1 r| + \cdots + \left|\tfrac{b_1 r}{b_0}\right| |c_{n-1} r^{n-1}| \\
&\le M + M^2 + M^2(1+M) + M^2(1+M)^2 + M^2(1+M)^3 + \cdots + M^2(1+M)^{n-1} \\
&= M + M^2\left(\frac{(1+M)^n - 1}{M}\right) = M(1+M)^n\,.
\end{aligned}
$$

Bu induction, we have $|c_n r^n| \le M(1+M)^n$ for all $n \ge 0$. Let $J_1 = \left(a - \frac{r}{1+M}, a + \frac{r}{1+M}\right)$. Let $x \in J_1$ so $|x - a| < \frac{r}{1+M}$. Then for all n we have

$$
|c_n (x-a)^n| = |c_n r^n| \cdot \frac{1}{(1+M)^n} \cdot \left|\frac{x-a}{r/(1+M)}\right|^n \le M \left|\frac{x-a}{r/(1+M)}\right|^n
$$

and so $\sum |c_n (x-a)^n|$ converges by the Comparison Test.

Note that from the definition of c_n we have $a_n = \sum\limits_{k=0}^{n} c_k b_{n-k}$, and so by multiplying power series, we have

$$
\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n (x-a)^n\right) = \sum_{n=0}^{\infty} a_n (x-a)^n
$$

for all $x \in I \cap J_1$. Finally note that $f(x) = \sum\limits_{n=0}^{\infty} b_n (x-a)^n$ is continuous in I and we have $f(0) = b_0 \neq 0$, and so there is an interval $J \subset I \cap J_1$ with $a \in J$ such that $f(x) \neq 0$ in J.

4.30 Theorem: *(Composition of Power Series) Let* $f(x) = \sum\limits_{n=0}^{\infty} a_n (x-a)^n$ *in an open interval* I *with* $a \in I$, *and let* $g(y) = \sum\limits_{m=0}^{\infty} b_m (y-b)^m$ *in an open interval* J *with* $b \in J$ *and with* $a_0 \in J$. *Let* K *be an open interval with* $a \in K$ *such that* $f(K) \subset J$. *For each* $m \ge 0$, *let* $c_{n,m}$ *be the coefficients, found by multiplying power series, such that* $\sum\limits_{n=0}^{\infty} c_{n,m} (x-a)^n = b_n \left(\sum\limits_{n=0}^{\infty} a_n (x-a)^n - b\right)^m$. *Then* $\sum\limits_{m \ge 0} c_{n,m}$ *converges for all* $m \ge 0$, *and for all* $x \in K$, $\sum\limits_{n \ge 0} \Big(\sum\limits_{m=0}^{\infty} c_{n,m}\Big)(x-a)^n$ *converges and*

$$
\sum_{n=0}^{\infty} \Big(\sum_{m=0}^{\infty} c_{n,m}\Big)(x-a)^n = g\big(f(x)\big)\,.
$$

Proof: We shall omit the proof. The proof would be fairly long and technical unless we first introduced some additional machinery. Both the proof and the statement of the theorem would become more elegant if we first defined and studied differentiation of complex-valued functions of a complex variable. This is done, for example, in PMATH 352.

4.31 Theorem: *(Integration of Power Series) Suppose that* $\sum a_n(x-a)^n$ *converges in the interval* I. *Then for all* $x \in I$, *the sum* $f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$ *is integrable on* $[a,x]$ *(or* $[x,a]$*) and*

$$\int_a^x \sum_{n=0}^{\infty} a_n(t-a)^n\,dt = \sum_{n=0}^{\infty}\int_a^x a_n(t-a)^n\,dt = \sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-a)^{n+1}.$$

Proof: This follows from uniform convergence.

4.32 Theorem: *(Differentiation of Power Series) Suppose that* $\sum a_n(x-a)^n$ *converges in the open interval* I. *Then the sum* $f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$ *is differentiable in* I *and*

$$f'(x)=\sum_{n=1}^{\infty} n\,a_n(x-a)^{n-1}.$$

Proof: We claim that the radius of convergence of $\sum a_n(x-a)^n$ is equal to the radius of convergence of $\sum na_n(x-a)^{n-1}$. Let R be the radius of convergence of $\sum a_n(x-a)^n$ and let S be the radius of convergence of $\sum na_n(x-a)^{n-1}$. Fix $x\in(a-R,a+R)$ so $|x-a|<R$ and $\sum\left|a_n(x-a)^n\right|$ converges. Choose r,s with $|x-a|<r<s<R$. Since $\lim\limits_{n\to\infty}\dfrac{(r/s)^n}{(1/n)}=0$, we can choose N so that $n\geq N \implies \left(\frac{r}{s}\right)^n<\frac{1}{n}$. Then for $n\geq N$ we have

$$\left|na_n(x-a)^n\right| = \left|n\left(\tfrac{r}{s}\right)^n\left(\tfrac{x-a}{r}\right)^n a_n s^n\right| \leq 1\cdot 1\cdot|a_n s^n|\,.$$

Since $\sum|a_ns^n|$ converges, $\sum\left|na_n(x-a)^n\right|$ converges by the Comparison Test, and so $\sum\left|na_n(x-a)^{n-1}\right|$ converges by Linearity. Thus $R\leq S$.

Now fix $x\in(a-S,a+s)$ so that $|x-a|<S$ and $\sum\left|na_n(x-a)^{n-1}\right|$ converges. Then $\sum\left|na_n(x-a)^n\right|$ converges by Linearity, and $\left|a_n(x-a)^n\right|\leq\left|na_n(x-a)^n\right|$ so $\sum\left|a_n(x-a)^n\right|$ converges by Comparison. Thus $S\leq R$ and so $R=S$ as claimed.

The theorem now follows from the uniform convergence of $\sum na_n(x-a)^{n-1}$.

4.33 Example: We have $\frac{1}{1+x}=\sum\limits_{n=0}^{\infty}(-1)^nx^n$ for $|x|<1$. By Integration of Power Series, $\ln x=\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}=\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n}x^n$ for $|x|<1$. In particular, we can take $x=\frac{1}{2}$ to get $\ln\frac{3}{2}=\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 2^n}$ and we can take $x=-\frac{1}{2}$ to get $\ln\frac{1}{2}=\sum\limits_{n=1}^{\infty}\frac{-1}{n\cdot 2^n}$, that is $\ln 2=\sum\limits_{n=1}^{\infty}\frac{1}{n\cdot 2^n}$.

Let us also argue that we can also take $x=1$. Note that the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ diverges when $x=-1$ (by the Integral Test) and converges when $x=1$ (by the Alternating Series Test), so the interval of convergence is $(-1,1]$. Thus the sum $f(x)=\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ is defined for $-1<x\leq 1$. We know already that $f(x)=\ln(1+x)$ for $-1<x<1$. By Abel's Theorem, the series converges uniformly on $[0,1]$, so by the Continuity of Power Series Theorem, the sum $f(x)=\sum\limits_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ is continuous on $[0,1]$ and in particular $f(x)$ is continuous at $x=1$. Since $f(x)=\ln(1+x)$ for $|x|<1$ and and since both $f(x)$ and $\ln(1+x)$ are continuous at 1 it follows that $f(1)=\ln 2$. Thus we have $\ln 2=\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$.

Taylor Series

4.34 Theorem: *Suppose that* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *in an open interval* I *centred at* a. *Then* f *is infinitely differentiable at* a *and for all* $n \geq 0$ *we have*

$$a_n = \frac{f^{(n)}(a)}{n!}\,,$$

where $f^{(n)}(a)$ *denotes the* n^{th} *derivative of* f *at* a.

Proof: By repeated application of the Differentiation of Power Series Theorem, for all $x \in I$, we have $f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$, $f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-a)^{n-2}$ and $f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)a_n(x-a)^{n-3}$, and in general

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-a)^{n-k}$$

and so $f(a) = a_0$, $f'(a) = a_1$, $f''(a) = 2\cdot 1\,a_2$ and $f'''(a) = 3\cdot 2\cdot 1\,a_3$, and in general

$$f^{(n)}(a) = n!\;a_n$$

4.35 Definition: Given a function $f(x)$ whose derivatives of all order exist at $x = a$, we define the **Taylor series** of $f(x)$ centered at a to be the power series

$$T(x) = \sum_{n\geq 0} a_n(x-a)^n \quad \text{where } a_n = \frac{f^{(n)}(a)}{n!}$$

and we define the l^{th} **Taylor Polynomial** of $f(x)$ centered at a to be the l^{th} partial sum

$$T_l(x) = \sum_{n=0}^{l} a_n(x-a)^n \quad \text{where } a_n = \frac{f^{(n)}(a)}{n!}$$

4.36 Example: Find the Taylor series centered at 0 for $f(x) = e^x$.

Solution: We have $f^{(n)}(x) = e^x$ for all n, so $f^{(n)}(0) = 1$ and $a_n = \frac{1}{n!}$ for all $n \geq 0$. Thus the Taylor series is

$$T(x) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\,x^n = 1 + x + \tfrac{1}{2!}x^2 = \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \cdots\,.$$

4.37 Example: Find the Taylor series centered at 0 for $f(x) = \sin x$.

Solution: We have $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f''''(x) = \sin x$ and so on, so that in general $f^{(2n)}(x) = (-1)^n \sin x$ and $f^{(2n+1)}(x) = (-1)^n \cos x$. It follows that $f^{(2n)}(0) = 0$ and $f^{(2n+1)}(0) = (-1)^n$, so we have $a_{2n} = 0$ and $a_{2n+1} = \frac{(-1)^n}{(2n+1)!}$. Thus

$$T(x) = \sum_{n=0}^{\infty} \tfrac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots\,.$$

4.38 Example: Find the Taylor series centered at 0 for $f(x) = (1+x)^p$ where $p \in \mathbb{R}$.

Solution: $f'(x) = p(1+x)^{p-1}$, $f''(x) = p(p-1)(1+x)^{p-2}$, $f'''(x) = p(p-1)(p-2)(1+x)^{p-3}$, and in general

$$f^{(n)}(x) = p(p-1)(p-2)\cdots(p-n+1)(1+x)^{p-n}\,,$$

so $f(0) = 1$, $f'(0) = p$, $f''(0) = p(p-1)$, and in general $f^{(n)}(0) = p(p-1)(p-2)\cdots(p-n+1)$, and so we have $a_n = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$. Thus the Taylor series is

$$T(x) = \sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \tfrac{p(p-1)}{2!}x^2 + \tfrac{p(p-1)(p-2)}{3!}x^3 + \tfrac{p(p-1)(p-2)(p-3)}{4!}x^4 + \cdots$$

where we use the notation

$$\binom{p}{0} = 1\ \text{, and for } n \geq 1,\ \binom{p}{n} = \tfrac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$$

4.39 Theorem: *(Taylor) Let $f(x)$ be infinitely differentiable in an open interval I with $a \in I$. Let $T_l(x)$ be the l^{th} Taylor polynomial for $f(x)$ centered at a. Then for all $x \in I$ there exists a number c between a and x such that*

$$f(x) - T_l(x) = \frac{f^{(l+1)}(c)}{(l+1)!}(x-a)^{l+1}\,.$$

Proof: When $x = a$ both sides of the above equation are 0. Suppose that $x > a$ (the case that $x < a$ is similar). Since $f^{(l+1)}$ is differentiable and hence continuous, by the Extreme Value Theorem it attains its maximum and minimum values, say M and m. Since $m \leq f^{(l+1)}(t) \leq M$ for all $t \in I$, we have

$$\int_a^{t_1} m\,dt \leq \int_a^{t_1} f^{(l+1)}(t)\,dt \leq \int_a^{t_1} M\,dt$$

that is

$$m(t_1 - a) \leq f^{(l)}(t_1) - f^{(l)}(a) \leq M(t_1 - a)$$

for all $t_1 > a$ in I. Integrating each term with respect to t_1 from a to t_2, we get

$$\tfrac{1}{2}m(t_2-a)^2 \leq f^{(l-1)}(t_2) - f^{(l)}(a)(t_2-a) \leq \tfrac{1}{2}M(t_t-a)^2$$

for all $t_2 > a$ in I. Integrating with respect to t_2 from a to t_3 gives

$$\tfrac{1}{3!}m(t_3-a)^3 \leq f^{(l-2)}(t_3) - f^{(l-2)}(a) - \tfrac{1}{2}f^{(l)}(a)(t_3-a)^3 \leq \tfrac{1}{3!}M(t_3-a)^3$$

for all $t_3 > a$ in I. Repeating this procedure eventually gives

$$\tfrac{1}{(l+1)!}m(t_{l+1}-a)^{l+1} \leq f(t_{l+1}) - T_l(t_{l+1}) \leq \tfrac{1}{(l+1)!}M(t_{l+1}-a)^{l+1}$$

for all $t_{l+1} > a$ in I. In particular $\frac{1}{(l+1)!}m(x-a)^{l+1} \leq f(x) - T_l(x) \leq \frac{1}{(l+1)!}M(x-a)^{l+1}$, so

$$m \leq \big(f(x) - T_l(x)\big)\tfrac{(l+1)!}{(x-a)^{l+1}} \leq M\,.$$

By the Intermediate Value Theorem, there is a number $c \in [a,x]$ such that

$$f^{(l+1)}(c) = \big(f(x) - T_l(x)\big)\frac{(l+1)!}{(x-a)^{l+1}}$$

.

4.40 Theorem: *The functions e^x, $\sin x$ and $(1+x)^p$ are all exactly equal to the sum of their Taylor series centered at 0 in the interval of convergence.*

Proof: First let $f(x)=e^x$ and let $x\in\mathbb{R}$. By Taylor's Theorem, $f(x)-T_l(x)=\dfrac{e^c x^{l+1}}{(l+1)!}$ for some c between 0 and x, and so

$$\big|f(x)-T_l(x)\big|\le\frac{e^{|x|}|x|^{l+1}}{(l+1)!}\,.$$

Since $\sum\dfrac{e^{|x|}|x|^{l+1}}{(l+1)!}$ converges by the Ratio Test, we have $\lim\limits_{l\to\infty}\dfrac{e^{|x|}|x|^{l+1}}{(l+1)!}=0$ by the Divergence Test, so $\lim\limits_{l\to\infty}\big(f(x)-T_l(x)\big)=0$, and so $f(x)=\lim\limits_{l\to\infty}T_l(x)=T(x)$.

Now let $f(x)=\sin x$ and let $x\in\mathbb{R}$. By Taylor's Theorem, $f(x)-T(x)=\dfrac{f^{(l+1)}(c)\,x^{l+1}}{(l+1)!}$ for some c between 0 and x. Since $f^{(l+1)}(x)$ is one of the functions $\pm\sin x$ or $\pm\cos x$, we have $\big|f^{(l+1)}(c)\big|\le 1$ for all c and so

$$\big|f(x)-T(x)\big|\le\frac{|x|^{l+1}}{(l+1)!}\,.$$

Since $\sum\dfrac{|x|^{l+1}}{(l+1)!}$ converges by the Ratio Test, $\lim\limits_{l\to\infty}\dfrac{|x|^{l+1}}{(l+1)!}=0$ by the Divergence Test, and so we have and $f(x)=T(x)$ as above.

Finally, let $f(x)=(1+x)^p$. The Taylor series centered at 0 is

$$T(x)=1+px+\tfrac{p(p-1)}{2!}x^2+\tfrac{p(p-1)(p-2)}{3!}x^3+\tfrac{p(p-1)(p-2)(p-3)}{4!}x^4+\cdots$$

and it converges for $|x|<1$. Differentiating the power series gives

$$T'(x)=p+\tfrac{p(p-1)}{1!}x+\tfrac{p(p-1)(p-2)}{2!}x^2+\tfrac{p(p-1)(p-2)(p-3)}{3!}x^3+\cdots$$

and so

$$\begin{aligned}(1+x)T'(x)&=p+\left(p+\tfrac{p(p-1)}{1!}\right)x+\left(\tfrac{p(p-1)}{1!}+\tfrac{p(p-1)(p-2)}{2!}\right)x^2\\&\qquad+\left(\tfrac{p(p-1)(p-2)}{2!}-\tfrac{p(p-1)(p-2)(p-3)}{3!}\right)x^3+\cdots\\&=p+\tfrac{p\cdot p}{1!}x+\tfrac{p\cdot p(p-1)}{2!}x^2+\tfrac{p\cdot p(p-1)(p-2)}{3!}x^3+\cdots\\&=p\,T(x)\,.\end{aligned}$$

Thus we have $(1+x)T'(x)=pT(x)$ with $T(0)=1$. This DE is linear since we can write it as $T'(x)-\frac{p}{1+x}T(x)=0$. An integrating factor is $\lambda=e^{\int-\frac{p}{1+x}\,dx}=e^{-p\ln(1+x)}=(1+x)^{-p}$ and the solution is $T(x)=(1+x)^{-p}\displaystyle\int 0\,dx=b(1+x)^p$ for some constant b. Since $T(0)=1$ we have $b=1$ and so $T(x)=(1+x)^p=f(x)$.