Chapter 7. Jordan Content and Integration

7.1 Definition: A (closed, n-dimensional) **rectangle** in $\mathbb{R}^n$ is a set of the form

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] = \left\{ x \in \mathbb{R}^n \,\middle|\, a_j \le x_j \le b_j \text{ for each index } j \right\}$$

where each $a_j, b_j \in \mathbb{R}$ with $a_j < b_j$. The **size** of the above rectangle R is

$$|R| = \prod_{j=1}^{n} (b_j - a_j).$$

A **partition** X of the above rectangle R consists of a partition $X_j = \{x_{j,0}, x_{j,1}, \cdots, x_{j,\ell_j}\}$ with

$$a_j = x_{j,0} < x_{j,1} < \cdots < x_{j,\ell_k} = b_j$$

for each index j. The above partition X divides the rectangle R into **sub-rectangles** R_k, where $k = (k_1, k_2, \cdots, k_n) \in \mathbb{R}^n$ with $1 \le k_j \le \ell_j$ for each index j, and where

$$R_k = [x_{1,k_1-1}, x_{1,k_1}] \times [x_{2,k_2-1}, x_{2,k_2}] \times \cdots \times [x_{n,k_n-1}, x_{n,k_n}].$$

If Y is another partition, given by $Y_j = \{y_{j,0}, \cdots, y_{j,m_j}\}$, then we say that Y is **finer** than X (or that X is **coarser** than Y) when $X_j \subseteq Y_j$ for each index j.

7.2 Example: Note that a 1-dimensional rectangle in $\mathbb{R}^1$ is a line segment and its size is its length, a 2-dimensional rectangle in $\mathbb{R}^2$ is a rectangle and its size is its area, and a 3-dimensional rectangle in $\mathbb{R}^3$ is a rectangular box and its size is its volume.

7.3 Note: When R is a rectangle in $\mathbb{R}^n$ and X and Y are any two partitions of R, the partition Z given by $Z_j = X_j \cup Y_j$ is finer that both X and Y.

7.4 Note: When R is a rectangle in $\mathbb{R}^n$ and X is a partition given by $X_j = \{x_{j,0}, \cdots, x_{j,\ell_j}\}$, then letting $K = K(X) = \left\{ k \in \mathbb{Z}^n \,\middle|\, 1 \le k_j \le \ell_j \text{ for all } j \right\}$, we have

$$\begin{aligned}
\sum_{k \in K} |R_k| &= \sum_{1 \le k_1 \le \ell_1} \sum_{1 \le k_2 \le \ell_2} \cdots \sum_{1 \le k_n \le \ell_n} \prod_{j=1}^{n} (x_{j,k_j} - x_{j,k_j-1}) \\
&= \prod_{j=1}^{n} \sum_{1 \le k_j \le \ell_j} (x_{j,k_j} - x_{j,k_j-1}) = \prod_{j=1}^{n} (x_{j,\ell_j} - x_{j,0}) \\
&= \prod_{j=1}^{n} (b_j - a_j) = |R|.
\end{aligned}$$

7.5 Definition: Let $A \subseteq \mathbb{R}^n$ be bounded. For a partition X of a rectangle R with $A \subseteq R$, we define the **upper** (or **outer**) **volume estimate of** A **with respect to** X, and the **lower** (or **inner**) **volume estimate of** A **with respect to** X, to be

$$U(A,X) = \sum_{R_k \cap \overline{A} \neq \emptyset} |R_k| = \sum_{k \in I} |R_k| \quad \text{and} \quad L(A,X) = \sum_{R_k \subseteq A^o} |R_k| = \sum_{k \in J} |R_k|$$

where $I = I(A,X) = \left\{k \in K \,|\, R_k \cap \overline{A} \neq \emptyset\right\}$ and $J = J(A,X) = \left\{k \in K \,\middle|\, R_k \subseteq A^o\right\}$ with $K = K(X) = \left\{k \in \mathbb{Z}^n \,\middle|\, 1 \leq k_j \leq \ell_j \text{ for each } j\right\}$.

7.6 Theorem: *(Basic Properties of Upper and Lower Volume Estimates) Let $A \subseteq \mathbb{R}^n$ be bounded, let R be a rectangle in $\mathbb{R}^n$ with $A \subseteq R$, and let X and Y be partitions of R.*
(1) If Y is finer than X then $0 \leq L(A,X) \leq L(A,Y) \leq U(A,Y) \leq U(A,X) \leq |R|$.
(2) $0 \leq L(A,X) \leq U(A,Y) \leq |R|$.
(3) $U(A,X) - L(A,X) = U(\partial A, X)$.

Proof: To prove Part 1, suppose that Y is finer than X. Note that each of the sub-rectangles R_k for the partition X is itself further partitioned into smaller sub-rectangles which are sub-rectangles for the partition Y, and denote these smaller sub-rectangles by $S_{k,1}, \cdots, S_{k,m_k}$. Then we have

$$U(A,X) = \sum_{k \in I} |R_k| \ \text{ and } \ U(A,Y) = \sum_{k \in I} \sum_{j \in J_k} |S_{k,j}|$$

where I is the set of $k \in K(X)$ such that $R_k \cap \overline{A} \neq \emptyset$ and J_k is the set of $j \in \{1, 2, \cdots, m_j\}$ such that $S_{k,j} \cap \overline{A} \neq \emptyset$. By Note 7.4, we have $\sum_{j=1}^{m_k} |S_{k,j}| = |R_k|$, and so

$$U(A,Y) = \sum_{k \in I} \sum_{j \in J_k} |S_{k,j}| \leq \sum_{k \in I} \sum_{j=1}^{m_j} |S_{k,j}| = \sum_{k \in I} |R_k| = U(A,X).$$

and also $U(A,X) = \sum_{k \in I} |R_k| \leq \sum_{k \in K(X)} |R_k| = |R|$. Thus we have $U(A,Y) \leq U(A,X) \leq |R|$. The proof that $L(A,X) \leq L(A,Y)$ is similar, and it is clear that $0 \leq L(A,X)$ and easy to see that $L(A,Y) \leq U(A,Y)$.

Note that Part 2 follows from Part 1 because, given any partitions X and Y for R, we can choose a partition Z which is finer than both X and Y, and then we have

$$0 \leq L(A,X) \leq L(A,Z) \leq U(A,Z) \leq U(A,Y) \leq |R|.$$

Finally, to prove Part 3, note that

$$U(A,X) - L(A,X) = \sum_{k \in L} |R_k| \ \text{ and } \ U(\partial A, X) = \sum_{k \in M} |R_k|$$

where L is the set of indices $k \in K(X)$ such that $R_k \cap \overline{A} \neq \emptyset$ and $R_k \not\subseteq A^o$, and M is the set of indices $k \in K(X)$ such that $R_k \cap \partial A \neq \emptyset$ (since ∂A is closed so that $\overline{\partial A} = \partial A$). We shall show that $K = M$. When $A = \emptyset$ we have $K = M = \emptyset$, so suppose $A \neq \emptyset$. If $k \in L$, that is if $R_k \cap \overline{A} \neq \emptyset$ and $R_k \not\subseteq A^o$ then we must have $R_k \cap \partial A \neq \emptyset$ because R_k is connected (indeed, if we had $R_k \cap \partial A = \emptyset$ then R_k would be separated by the disjoint nonempty open sets A^o and $\overline{A}^c$: note that we have $A^o \neq \emptyset$ because $R_k \cap \overline{A} \neq \emptyset$, and we have $\overline{A}^c \neq \emptyset$ because $R_k \not\subseteq A^o$) and hence $L \subseteq M$. If $k \in M$, that is if $R_k \cap \partial A \neq \emptyset$ then, since $\partial A \subseteq \overline{A}$ we have $R_k \cap \overline{A} \neq \emptyset$, and since A^o and ∂A are disjoint we have $R_k \not\subseteq A^o$, and hence $k \in M$. Thus $K = M$, as required.

7.7 Definition: Let $A \subseteq \mathbb{R}^n$ be bounded. We define the **upper** (or **outer**) **volume** (or **Jordan content**), and the **lower** (or **inner**) **volume** (or **Jordan content**), of A to be

$$U(A) = \inf\{U(A,X) \mid X \text{ is a partition of some rectangle } R \text{ with } A \subseteq R\}$$
$$L(A) = \sup\{L(A,X) \mid X \text{ is a partition of some rectangle } R \text{ with } A \subseteq R\}.$$

7.8 Theorem: *(Basic Properties of Upper and Lower Volumes) Let $A \subseteq \mathbb{R}^n$ be bounded.*
(1) If R is any rectangle with $A \subseteq R$ then $U(A) = \inf\{U(A,X) \mid X \text{ is a partition of } R\}$.
(2) $U(A) - L(A) = U(\partial A)$.

Proof: Given a rectangle R with $A \subseteq R$, let $U_R(A) = \inf\{U(A,X) \mid X \text{ is a partition of } R\}$. To prove Part 1, it suffices to show that for any two rectangles R, S in $\mathbb{R}^n$ which contain A, we have $U_R(A) = U_S(A)$. Let R and S be rectangles in $\mathbb{R}^n$ which contain A, say $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $S = [c_1, d_1] \times \cdots \times [c_n, d_n]$.

Suppose first that $R \subseteq S$ with $c_j < a_j$ and $b_j < d_j$. Given any partition Y of S, we can extend Y to a finer partition Z of S by adding the endpoints of R, that is by letting $Z_j = Y_j \cup \{a_j, b_j\}$, and then we can restrict Z to a partition X of R as follows: if, for a fixed index j, we have $Z_j = \{z_0, \cdots, z_k, \cdots, z_\ell, \cdots, z_m\}$ with $z_0 = c_j$, $z_k = a_j$, $z_\ell = b_j$ and $z_m = d_j$, then we take $X_j = \{z_k, \cdots, z_\ell\}$. Then we have $U(A,X) \leq U(A,Z) \leq U(A,Y)$. Since for every partition Y of S there exists a corresponding partition X of R for which $U(A,X) \leq U(A,Y)$, it follows that

$$\inf\{U(A,X) \mid X \text{ is a partition of } R\} \leq \inf\{U(A,Y) \mid Y \text{ is a partition of } S\},$$

that is $U_R(A) \leq U_S(A)$. Now let $\epsilon > 0$ and suppose that we are given a partition X of R. Choose s_j and t_j with $c_j < s_j < a_j$ and $b_j < t_j < b_j$ so that for the rectangle $T = [s_1, t_1] \times \cdots \times [s_n, t_n]$ we have $|T| - |R| \leq \epsilon$. Extend the partition X of R to the partition Y of S by adding the endpoints of S and T, that is by letting $Y_j = X_j \cup \{c_j, s_j, t_j, d_j\}$. Note that the sub-rectangles of S which intersect with $\overline{A}$ include all of the sub-rectangles of R which intersect with $\overline{A}$ together with some of the sub-rectangles which lie in T but not R, and so we have $U(A,Y) \leq U(A,X) + |T| - |R| \leq U(A,X) + \epsilon$. Since for each partition X of R there is a corresponding partition Y of S for which $U(A,Y) \leq U(A,X) + \epsilon$, it follows that

$$\inf\{U(A,Y) \mid Y \text{ is a partition of } S\} \leq \inf\{U(A,X) \mid X \text{ is a partition of } R\} + \epsilon,$$

that is $U_S(A) \leq U_R(A) + \epsilon$, and since $\epsilon > 0$ was arbitrary, it follows that $U_S(A) \leq U_R(A)$. Thus we have proven that $U_R(A) = U_S(A)$ in the case that $R \subseteq S$ with $c_j < a_j < b_j < d_j$.

In the general case that $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $S = [c_1, d_1] \times \cdots \times [c_n, d_n]$ are any rectangles which both contain A, we can choose a rectangle $T = [s_1, t_1] \times \cdots \times [s_n, t_n]$ with $s_j < \min\{a_j, c_j\}$ and $t_j > \max\{b_j, d_j\}$, and then we can apply the result of the above paragraph to obtain $U_R(A) = U_T(A) = U_S(A)$, as required, proving Part 1.

Let us prove Part 2. Given any partition X of any rectangle R containing A, we have $U(A) - L(A) \leq U(A,X) - L(A,X) = U(\partial A, X)$, and hence (by taking the infemum on both sides) $U(A) - L(A) \leq U(\partial A)$. It remains to show that $U(A) - L(A) \geq U(\partial A)$. Let $\epsilon > 0$. Choose a rectangle R containing A, and choose a partition X of R such that $L(A) - \epsilon < L(A,X) \leq L(A)$. By Part 1, we can choose a partition Y of the same rectangle R such that $U(A) \leq U(A,Y) < U(A) + \epsilon$. Let Z be a partition of R which is finer than both X and Y. Then we have $L(A) - \epsilon < L(A,X) \leq L(A,Z)$ and $U(A,Z) \leq U(A,Y) < U(A) + \epsilon$ and hence $U(\partial A) \leq U(\partial A, Z) = U(A,Z) - L(A,Z) < U(A) - L(A) + 2\epsilon$. Since $\epsilon > 0$ was arbitrary, we have $U(\partial A) \leq U(A) - L(A)$, as required.

7.9 Theorem: *For bounded sets $A, B \subseteq \mathbb{R}^n$, we have $U(A \cup B) \leq U(A) + U(B)$.*

Proof: First we note that for any sets $A, B \subseteq \mathbb{R}^n$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$: Indeed, since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ we have $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$ so that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. On the other hand, since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, we have $A \cup B \subseteq \overline{A} \cup \overline{B}$ and so, since $\overline{A} \cup \overline{B}$ is closed, and contains $A \cup B$, it follows that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Let $A, B \subseteq \mathbb{R}^n$ be bounded. Let R be a rectangle which contains $A \cup B$. Let $\epsilon > 0$. Choose a partition X of R so that $U(A) \leq U(A, X) + \frac{\epsilon}{2}$ and $U(B) \leq U(B, X) \leq \frac{\epsilon}{2}$ (we can do this by Part 1 of Theorems 7.8 and 7.6: let Y be a partition of R such that $U(A) \leq U(A, Y) + \frac{\epsilon}{2}$ let Z be a partition of R such that $U(B) \leq U(B, Z) + \frac{\epsilon}{2}$, then let X be a partition finer than both Y and Z). Let $K = K(X)$, let $I(A \cup B) = I(A \cup B, X)$, $I(A) = I(A, X)$ and $I(B) = I(B, X)$, as in Definition 7.5. Since $\overline{A \cup B} = \overline{A} \cup \overline{B}$, for each index $k \in K$ we have

$$k \in I(A \cup B) \iff R_k \cap \overline{A \cup B} \neq \emptyset \iff (R_k \cap \overline{A}) \cup (R_k \cap \overline{B}) \neq \emptyset \iff \big(k \in I(A) \text{ or } k \in I(B)\big),$$

$$U(A \cup B, X) = \sum_{k \in I(A \cup B)} |R_k| \leq \sum_{k \in I(A)} |R_k| + \sum_{k \in I(B)} |R_k| = U(A, X) + U(B, X) \leq U(A) + U(B) + \epsilon.$$

Since $U(A \cup B, X) \leq U(A) + U(B) + \epsilon$ for all partitions X of R, it follows (from Part 1 of Theorem 7.8) that $U(A \cup B) \leq U(A) + U(B) + \epsilon$, and since $\epsilon > 0$ was arbitrary, it follows that $U(A \cup B) \leq U(A) + U(B)$, as required.

7.10 Definition: Let $A \subseteq \mathbb{R}^n$ be bounded. We say that A **has well-defined volume** (or **Jordan content**), or that A is **Jordan measurable**, or that A is a **Jordan region**, when $U(A) = L(A)$, or equivalently (by Part 2 of Theorem 7.8) when $U(\partial A) = 0$. In this case, we define the (n-dimensional) **volume** of A (or the **Jordan content**) of A to be

$$\text{Vol}(A) = U(A) = L(A).$$

7.11 Theorem: *Every rectangle R in $\mathbb{R}^n$ is Jordan measurable with* $\text{Vol}(R) = |R|$.

Proof: Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a ractangle in $\mathbb{R}^n$. By Note 7.4, we have $U(R, X) = |R|$ for every partition X of R, so by Part 1 of Theorem 7.8, it follows that $U(R) = |R|$. By Part 2 of Theorem 7.8, we have $U(R) - L(R) = U(\partial R) \geq 0$ so that $L(R) \leq U(R)$. Let $\epsilon > 0$. Choose a rectangle S of the form $S = [c_1, d_1] \times \cdots \times [c_n, d_n]$ with $a_1 < c_1$ and $d_1 < b_1$ (so that $S \subseteq R^o$) such that $|R| - |S| < \epsilon$. Let X be the partition of R given by $X_j = \{a_j, c_j, d_j, b_j\}$. Since S is a sub-rectangle for this partition with $S \subseteq R^o$ we have $L(R, X) \geq |S|$, and so $L(R) \geq L(R, X) \geq |S| > |R| - \epsilon$. Since $\epsilon > 0$ was arbitrary, it follows that $L(R) \geq |R|$. Thus we have $L(R) = |R| = U(R)$.

7.12 Theorem: *(Properties of Jordan Content) Let $A, B \subseteq \mathbb{R}^n$ be Jordan measurable.*
(1) If $A \subseteq B$ then $\text{Vol}(A) \leq \text{Vol}(B)$.
(2) A^o and $\overline{A}$ are Jordan measurable with $\text{Vol}(A^o) = \text{Vol}(A) = \text{Vol}(\overline{A})$.
(3) $A \cup B$, $A \cap B$ and $A \setminus B$ are Jordan measurable with $\text{Vol}(A \setminus B) = \text{Vol}(A) - \text{Vol}(A \cap B)$ *and* $\text{Vol}(A \cup B) = \text{Vol}(A) + \text{Vol}(B) - \text{Vol}(A \cap B)$. *If $A \cap B = \emptyset$ then* $\text{Vol}(A \cup B) = \text{Vol}(A) + \text{Vol}(B)$.

Proof: To prove Part 1, suppose that $A \subseteq B$. Let R be a rectangle containing B and let X be a partition of R into the sub-rectangles R_k with $k \in K(X)$. Since $A \subseteq B$, we have $\overline{A} \subseteq \overline{B}$, so for $k \in K(X)$, if $R_k \cap \overline{A} \neq \emptyset$ then $R_k \cap \overline{B} \neq \emptyset$. This shows that $I(A, X) \subseteq I(B, X)$ and hence $U(A, X) = \sum_{k \in I(A,X)} |R_k| \leq \sum_{k \in I(B,X)} |R_k| = U(B, X)$. Since $U(A, X) \leq U(B, X)$ for every partition X of R, we have $U(A) \leq U(B)$ (by Part 1 of Theorem 7.8). Since A and B are measurable, this means that $\text{Vol}(A) \leq \text{Vol}(B)$, as required.

Let us prove Part 2. Since A^o is open we have $(A^o)^o = A^o$, and since $A^o \subseteq A$ we have $\overline{A^o} \subseteq \overline{A}$, and hence $\partial(A^o) = \overline{A^o} \setminus (A^o)^o = \overline{A^o} \setminus A^o \subseteq \overline{A} \setminus A^o = \partial A$. Since $\partial A^o \subseteq \partial A$ we have $U(\partial A^o) \le U(\partial A)$ (by Part 1), and since A is measurable we have $U(\partial A) = 0$. Thus $U(\partial A^o) = 0$ so that A^o is Jordan measurable. Similarly, we have $\overline{\overline{A}} = \overline{A}$ and $A^o \subseteq \overline{A}^o$ so that $\partial \overline{A} = \overline{\overline{A}} \setminus \overline{A}^o = \overline{A} \setminus \overline{A}^0 \subseteq \overline{A} \setminus A^o = \partial A$ and hence $U(\partial \overline{A}) \le U(\partial A) = 0$ so that $\overline{A}$ is Jordan measurable. Now let R be a rectangle containing A and let X be a partition of R. From the definition of $U(A,X)$ it is immediate that $U(A,X) = U(\overline{A},X)$, and from the definition of $L(A,X)$ it is immediate that $L(A,X) = L(A^o,X)$. Since this holds for all partitions X of R, we have $U(A) = U(\overline{A})$ and $L(A) = L(A^o)$. Since A is measurable, this gives $L(A^o) = L(A) = U(A) = U(\overline{A})$, and since A^o and $\overline{A}$ are measurable, this gives $\mathrm{Vol}(A^o) = \mathrm{Vol}(A) = \mathrm{Vol}(\overline{A})$, as required.

We move on to the proof of Part 3. To prove that $A \cup B$ is Jordan measurable, we note that $\partial(A \cup B) \subseteq \partial A \cup \partial B$: indeed, recall (as shown in the proof of Theorem 7.9) that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Also note that since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ we have $A^o \subseteq (A \cup B)^o$ and $B^o \subseteq (A \cup B)^o$ so that $A^o \cup B^o \subseteq (A \cup B)^o$. Thus

$$\begin{aligned} x \in \partial(A \cup B) &\Longrightarrow x \in \overline{A \cup B} \text{ and } x \notin (A \cup B)^o \\ &\Longrightarrow x \in \overline{A} \cup \overline{B} \text{ and } x \notin A^o \cup B^o \\ &\Longrightarrow \big(x \in \overline{A} \text{ and } x \notin A^o\big) \text{ and } \big(x \in \overline{B} \text{ and } x \notin B^o\big) \\ &\Longrightarrow x \in \partial A \cup \partial B. \end{aligned}$$

Since $\partial(A \cup B) \subseteq \partial A + \partial B$, Theorem 7.9 gives $U(\partial(A \cup B)) \le U(\partial A) + U(\partial B)$. Since A and B are Jordan measurable so that $U(\partial A) = 0$ and $U(\partial B) = 0$, we also have $U(\partial(A \cup B)) = 0$ so that $A \cup B$ is Jordan measurable. We can prove that $A \cap B$ and $A \setminus B$ are measuable in the same way, by showing that $\partial(A \cap B) \subseteq \partial A \cup \partial B$ and $\partial(A \setminus B) \subseteq \partial A \cup \partial B$, and we leave this as an exercise.

It remains to prove the various volume formulas. First, suppose that $A \cap B = \emptyset$. We know, from Theorem 7.9 that $U(A \cap B) \le U(A) + U(B)$. Let R be a rectangle which contains $A \cup B$, and let X be a partition of R such that $L(A,X) \ge L(A) - \frac{\epsilon}{2}$ and $L(B,X) \ge L(B) - \frac{\epsilon}{2}$. Since $A^o \subseteq A \subseteq A \cup B \subseteq \overline{A \cup B}$, it follows that if $k \in J(A^o, X)$, that is if $R_k \subseteq A^0$, then we have $R_k \subseteq \overline{A \cup B}$ so that $R_k \cap \overline{A \cup B} \ne \emptyset$, that is $k \in I(A \cap B, X)$, so we have $J(A,X) \subseteq I(A \cup B, X)$. Similarly, since $B^o \subseteq \overline{A \cup B}$, we have $J(B,X) \subseteq I(A \cup B, X)$. Also note that since $A \cap B = \emptyset$, we also have $A^o \cap B^o = \emptyset$, so it is not possible to have both $R_k \subseteq A^o$ and $R_k \subseteq B^o$, and it follows that $J(A,X) \cap J(B,X) = \emptyset$. Thus

$$U(A \cup B, X) = \sum_{k \in I(A \cap B, X)} |R_k| \ge \sum_{k \in J(A,X)} |R_k| + \sum_{k \in J(B,X)} |R_k| = L(A,X) + L(B,X) \ge L(A) + L(B) - \epsilon.$$

Since $U(A \cup B, X) \ge L(A) + L(B) - \epsilon$ for all partitions X of R, and since $\epsilon > 0$ was arbitrary, we have $U(A \cup B) \ge L(A) + L(B)$. Together with Theorem 7.9, this gives

$$L(A) + L(B) \le U(A \cup B) \le U(A) + U(B).$$

Since $L(A) = U(A) = \mathrm{Vol}(A)$ and $L(B) = U(B) = \mathrm{Vol}(B)$ and $U(A \cup B) = \mathrm{Vol}(A \cup B)$, we have proven that, if $A \cap B = \emptyset$ then $\mathrm{Vol}(A \cup B) = \mathrm{Vol}(A) + \mathrm{Vol}(B)$.

Finally, we note that the other two formulas (which apply whether or not A and B are disjoint), follow from the special case of disjoint sets: Indeed, the set A is the disjoint union $A = (A \setminus B) \cup (A \cap B)$, so we have $\mathrm{Vol}(A) = \mathrm{Vol}(A \setminus B) + \mathrm{Vol}(A \cap B)$, and $A \cup B$ is the disjoint union $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap H)$ so that $\mathrm{Vol}(A \cup B) = \mathrm{Vol}(A \setminus B) + \mathrm{Vol}(B \setminus A) + \mathrm{Vol}(A \cap B) = \mathrm{Vol}(A) + \mathrm{Vol}(B) - \mathrm{Vol}(A \cap B)$.

7.13 Definition: A **cube** in $\mathbb{R}^n$ is a rectangle $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ in $\mathbb{R}^n$ with equal side lengths, that is with $b_k - a_k = b_\ell - a_\ell$ for all $k \neq \ell$.

7.14 Theorem: *(Alternate Characterizations of Outer Jordan Content) Let $A \subseteq \mathbb{R}^n$ be bounded. Then*

$$U(A) = \inf\left\{ \sum_{j=1}^{m} |R_j| \,\middle|\, R_1, R_2, \cdots, R_m \text{ are rectangles } A \subseteq \bigcup_{j=1}^{m} R_j \right\}$$
$$= \inf\left\{ \sum_{j=1}^{m} |Q_j| \,\middle|\, Q_1, Q_2, \cdots, Q_m \text{ are cubes of equal size with } A \subseteq \bigcup_{j=1}^{m} Q_j \right\}.$$

Proof: Let

$$\mathcal{R} = \left\{ \sum_{R_k \cap \overline{A} \neq \emptyset}^{m} |R_k| \,\middle|\, X \text{ is a partition of some rectangle } R \text{ with } A \subseteq R \right\},$$
$$\mathcal{S} = \left\{ \sum_{j=1}^{m} |R_j| \,\middle|\, R_1, R_2, \cdots, R_m \text{ are rectangles with } A \subseteq \bigcup_{j=1}^{m} R_j \right\}, \text{ and}$$
$$\mathcal{T} = \left\{ \sum_{j=1}^{m} |Q_j| \,\middle|\, Q_1, Q_2, \cdots, Q_m \text{ are squares of equal size with } A \subseteq \bigcup_{j=1}^{m} Q_j \right\}.$$

and note that $U(A) = \inf \mathcal{R}$. We leave the proof that $U(A) = \inf \mathcal{S}$ as an exercise, and we prove that $U(A) = \inf \mathcal{T}$. When $Q_1, \cdots, Q_m$ are cubes of equal size with $A \subseteq \bigcup_{k=1}^{m} Q_k$, we know that $U(A) \leq \sum_{k=1}^{m} |Q_k|$ by Theorem 7.9, and hence $U(A) \leq \inf \mathcal{S}$. It remains to show that $\inf \mathcal{S} \leq U(A)$.

Let $\epsilon > 0$. Choose a rectangle R with $A \subseteq R$, and choose a partition X of R into sub-rectangles R_k such that $U(A, X) \leq U(A) + \frac{\epsilon}{2}$. Let $k_1, \cdots, k_m$ be the values of k for which $R_k \cap \overline{A} \neq \emptyset$, so we have $\overline{A} \subseteq \bigcup_{i=1}^{m} R_{k_i}$ and $\sum_{i=1}^{m} |R_{k_i}| = U(A, X) \leq U(A) + \frac{\epsilon}{2}$. For each index i, choose a rectangle S_i with $R_{k_i} \subseteq S_i$ such that the endpoints of all the component intervals of all the rectangles S_i are rational and $\sum_{i=1}^{m} |S_i| \leq \sum_{i=1}^{m} |R_{k_i}| + \frac{\epsilon}{2}$. Let d be a common denominator of all the endpoints of all the rectangles S_i, and partition each rectangle S_i into cubes $Q_{i,1}, Q_{i,2}, \cdots, Q_{i,\ell_i}$ all with sides of length $\frac{1}{d}$. Then we have $A \subseteq \bigcup_{i=1}^{m} S_i = \bigcup_{i=1}^{m} \bigcup_{j=1}^{\ell_i} Q_{i,j}$ and

$$\sum_{i=1}^{m} \sum_{j=1}^{\ell_i} |Q_{i,j}| = \sum_{i=1}^{m} |S_i| \leq \textstyle\sum_{i=1}^{m} |R_{k_i}| + \frac{\epsilon}{2} \leq U(A) + \epsilon.$$

Thus $\inf \mathcal{S} \leq U(A) + \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\inf \mathcal{S} \leq U(a)$, as required.

7.15 Definition: For a map $g : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^m$, we say that g is **Lipschitz continuous** on A when there is a constant $c \geq 0$ such that $|g(x) - g(y)| \leq c|x - y|$ for all $x, y \in A$, and we say that g is **open** when $g(U)$ is open in B for every open set U in A.

7.16 Theorem: *Let $A \subseteq \mathbb{R}^n$ be bounded and let $g : A \to \mathbb{R}^n$ be Lipschitz continuous.*

(1) If $U(A) = 0$ and g is Lipschitz continuous then $U(g(A)) = 0$.
(2) If A is Jordan measurable and g is open then $g(A)$ is Jordan measurable.

Proof: The proof is left as an exercise.

7.17 Definition: Let $A \subseteq \mathbb{R}^n$ be a Jordan region and let $f : A \to \mathbb{R}$ be a bounded function. Let X be a partition of a rectangle R in $\mathbb{R}^n$ which contains A, and let R_k, $k \in K$ be the sub-rectangles. Extend f to a function $g : R \to \mathbb{R}$ by defining $g(x) = f(x)$ when $x \in A$ and $g(x) = 0$ when $x \in R \setminus A$. The **upper Riemann sum** of f on A for the partition X and the **lower Riemann sum** of f on A for X are given by

$$U(f,X) = \sum_{k\in K} M_k\,|R_k| \;\text{ and }\; L(f,X) = \sum_{k\in K} m_k\,|R_k|$$

where $M_k = \sup\big\{g(x)\,|\,x \in R_k\big\}$ and $m_k = \inf\{f(x)\,|\,x \in R_k\}$. The **upper integral** of f on A and the **lower integral** of f on A are given by

$$U(f) = \inf\Big\{U(f,X)\,\Big|\; X \text{ is a partition of some rectangle } R \text{ with } A \subseteq R\Big\}$$
$$L(f) = \sup\Big\{L(f,X)\,\Big|\,X \text{ is a partition of some rectangle } R \text{ with } A \subseteq R\Big\}.$$

We say that f is (Riemann) **integrable** on A when $U(f) = L(f)$ and, in this case, we define the (Riemann) **integral** of f on A to be

$$\int_A f = \int_A f(x)\,dV = \int_A f(x_1,\cdots,x_n)\,dx_1dx_2\cdots dx_n = U(f) = L(f).$$

7.18 Theorem: *(Properties of Upper and Lower Riemann Sums) Let $A \subseteq \mathbb{R}^n$ be a Jordan region, let $f : A \to \mathbb{R}$ be a bounded function, let R be a rectangle which contains A, and let X and Y be two partitions of R.*
(1) If Y is finer than X then $L(f,X) \le L(f,Y) \le U(f,Y) \le U(f,X)$.
(2) We have $L(f,X) \le U(f,Y)$.

Proof: Let $g : R \to \mathbb{R}$ be the extension of f by zero. When $M_k = \sup\{g(x)\,|\,x\in R_k\}$ and $m_k = \inf\{g(x)\,|\,x\in R_k\}$, we have $m_k \le M_k$ for all $k \in K = K(X)$ so that

$$L(f,X) = \sum_{k\in K} m_k|R_k| \le \sum_{k\in K} M_k|R_k| = U(f,X).$$

Similarly, we have $L(f,Y) \le U(f,Y)$.

Suppose that Y is finer than X. Note that each of the sub-rectangles R_k for the partition X is itself further partitioned into smaller sub-rectangles which are sub-rectangles for the partition Y, and denote these smaller sub-rectangles by $S_{k,1},\cdots,S_{k,m_k}$. Note that $|R_k| = \sum_{j=1}^{m_k}|S_{k,j}|$ by Note 7.4. Let $M_k = \sup\{g(x)|\,x\in R_k\}$ and $N_{k,j} = \sup\{g(x)|\,x\in S_{k,j}\}$. Since $R_k = \bigcup_{j=1}^{m_k} S_{k,j}$, we have $M_k = \max\{N_{k,j}|1\le j\le m_k\}$ and hence

$$U(f,X) = \sum_{k\in K} M_k|R_k| = \sum_{k\in K}\sum_{j=1}^{m_k} M_k|S_{k,j}| \ge \sum_{k\in K}\sum_{j=1}^{m_k} N_{k,j}|S_{k,j}| = U(f,Y).$$

A similar argument shows that $L(f,X) \le L(f,Y)$. This completes the proof of Part 1.

Part 2 follows from Part 1. Indeed, given any partitions X and Y of R, we can choose a partition Z which is finer than both X and Y, and then we have

$$L(f,X) \le L(f,Z) \le U(f,Z) \le U(f,Y).$$

7.19 Theorem: *(Properties of Upper and Lower Integrals) Let $A \subseteq \mathbb{R}^n$ be a Jordan region, and let $f : A \to \mathbb{R}$ be a bounded function.*
(1) If R is any rectangle with $A \subseteq \mathbb{R}^n$ then $U(f) = \inf\big\{U(f,X)\,\big|\,X$ is a partition of $R\big\}$ and $L(f) = \sup\big\{L(f,X)\,\big|\,X$ is a partition of $R\big\}$.
(2) We have $L(f) \le U(f)$.

Proof: To prove Part 1, imitate the proof of Part 1 of Theorem 7.8. Part 2 follows from Part 1 of this theorem together with Part 2 of the previous theorem.

7.20 Theorem: *(Characterization of Integrability) Let $A \subseteq \mathbb{R}^n$ be a Jordan region, and let $f : A \to \mathbb{R}$ be a bounded function. Then f is integrable on A if and only if for every $\epsilon > 0$ there exits a partition X of a rectangle R with $A \subseteq R$ such that $U(f, X) - L(f, X) < \epsilon$.*

Proof: Suppose that f is integrable on A, so we have $U(f) = L(f)$. Let R be a rectangle with $A \subseteq R$. By Part 1 of Theorem 7.19, we can choose a partition Y of R such that $U(f, Y) < U(f) + \frac{\epsilon}{2}$, and we can choose a partition Z of R such that $L(f, Z) > L(f) - \frac{\epsilon}{2}$. Let X be a partition of R which is finer than both Y and Z. By Part 1 of Theorem 7.18, we have $U(f, X) \leq U(f, Y)$ and $L(f, X) \geq L(f, Z)$, and hence

$$U(f,X) - L(f,X) \leq U(f,Y) - L(f,Z) < \left(U(f) + \tfrac{\epsilon}{2}\right) - \left(L(f) - \tfrac{\epsilon}{2}\right) = U(f) - L(f) + \epsilon = \epsilon.$$

Suppose, conversely, that for every $\epsilon > 0$ there exists a partition X of a rectangle R with $A \subseteq R$ such that $U(f, X) - L(f, X) < \epsilon$. Let $\epsilon > 0$. Choose R and X so that $U(f, X) - L(f, X) < \epsilon$. By the definition of $U(f)$ and $L(f)$, we have $U(f) \leq U(f, X)$ and $L(f) \geq L(f, X)$, and so $U(f) - L(f) \leq U(f, X) - L(f, X) < \epsilon$. Since $U(f) - L(f) < \epsilon$ for every $\epsilon > 0$, it follows that $U(f) \leq L(f)$. On the other hand, we have $U(f) \geq L(f)$ by Part 2 of Theorem 7.19. Thus $U(f) = L(f)$ so that f is integrable.

7.21 Theorem: *(Continuity and Integrability) Let $A \subseteq \mathbb{R}^n$ be a Jordan region, and let $f : A \to \mathbb{R}$ be a bounded function. If f is uniformly continuous on A, then f is integrable.*

Proof: Suppose that f is bounded and uniformly continuous on A. Choose a rectangle R with $A \subseteq R$ and $|R| > 0$. Let $\epsilon > 0$. Since f is bounded, we can choose $M > 0$ so that $|f(x)| \leq M$ for all $x \in A$. Since f is uniformly continuous on A, we can choose $\delta > 0$ such that for all $x, y \in A$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\epsilon}{2|R|}$. Choose a partition X of R, into sub-rectangles R_k, which is fine enough so that firstly, we have $x, y \in R_k \implies |x - y| < \delta$ and, secondly, we have $U(\partial A, X) = \sum_{R_k \cap \partial A \neq \emptyset} |R_k| < \frac{\epsilon}{2M}$ (we can do this since $U(\partial A) = 0$). Since $\overline{A}$ is the disjoint union $\overline{A} = A^o \cup \partial A$, the rectangles R_k come in three varieties: $R_k \cap \overline{A} = \emptyset$, $R_k \cap \partial A \neq \emptyset$ or $R_k \subseteq A^o$. Let g be the extension of f by zero to R, and write $M_k = \sup\{g(x) | x \in R_k\}$ and $m_k = \inf\{g(x) | x \in R_k\}$. When $R_k \cap \overline{A} = \emptyset$, we have $g(x) = 0$ for all $x \in R_k$, and so

$$\sum_{R_k \cap \overline{A} = \emptyset} (M_k - m_k)|R_k| = 0.$$

When $R_k \cap \partial A \neq \emptyset$ we have $|g(x)| \leq M$ for all $x \in R_k$ so that

$$\sum_{R_k \cap \partial A \neq \emptyset} (M_k - m_k)|R_k| \leq 2M \sum_{R_k \cap \partial A \neq \emptyset} |R_k| < \tfrac{\epsilon}{2}.$$

When $R_k \subseteq A^o$, for all $x, y \in R_k$ we have $x, y \in A$ with $|x - y| < \delta$ so that $|g(x) - g(y)| = |f(x) - f(y)| < \frac{\epsilon}{2|R|}$, and hence $M_k - m_k \leq \frac{\epsilon}{2|R|}$ so that

$$\sum_{R_k \subseteq A^o} (M_k - m_k)|R_k| \leq \tfrac{\epsilon}{2|R|} \sum_{R_k \subseteq A^o} |R_k| \leq \tfrac{\epsilon}{2}.$$

Thus

$$U(f,X) - L(f,X) = \sum_{R_k \cap \overline{A} = \emptyset} (M_k - m_k)|R_k| + \sum_{R_k \cap \partial A \neq \emptyset} (M_k - m_k)|R_k| + \sum_{R_k \subseteq A^o} (M_k - m_k)|R_k| < \epsilon.$$

Thus f is integrable, by Theorem 7.20.

7.22 Theorem: *(Integration and Volume) If $A \subseteq \mathbb{R}^n$ is a Jordan region then*

$$\mathrm{Vol}(A) = \int_A 1\, dV.$$

Proof: Suppose that A is Jordan measurable, so we have $U(A) = L(A) = \mathrm{Vol}(A)$. Let R be a rectangle with $A \subseteq R$. Let $f : A \to \mathbb{R}$ be the constant function $f(x) = 1$, and let $g : R \to \mathbb{R}$ be the extension of f by zero. Choose a partition X of R, with sub-rectangles R_k, such that $U(A, X) \leq U(A) - \epsilon = \mathrm{Vol}(A) - \epsilon$ and $L(A, X) \geq L(A) - \epsilon = \mathrm{Vol}(A) - \epsilon$. Let $M_k = \sup\{g(x) | x \in R_k\}$ and $m_k = \inf\{g(x) | x \in R_k\}$. When $R_k \cap \overline{A} = \emptyset$ we have $g(x) = 0$ for all $x \in R_k$ so that $M_k = 0$, and for all k we have $M_k \leq 1$, and so

$$U(f) \leq U(f, X) = \sum_{R_k \cap \overline{A} \neq \emptyset} M_k |R_k| \leq \sum_{R_k \cap \overline{A} \neq \emptyset} |R_k| = U(A, X) \leq \mathrm{Vol}(A) + \epsilon.$$

When $R_k \subseteq A^o$ we have $g(x) = 1$ for all $x \in R_k$ so that $m_k = 1$, and for all k we have $m_k \geq 0$, and so

$$L(f) \geq L(f, X) \geq \sum_{R_k \subseteq A^o} m_k |R_k| = \sum_{R_k \subseteq A^o} |R_k| = L(A, X) \geq \mathrm{Vol}(A) - \epsilon.$$

Since $\mathrm{Vol}(A) - \epsilon \leq L(f) \leq U(f) \leq \mathrm{Vol}(A) + \epsilon$ for every $\epsilon > 0$, we have $U(f) = L(f) = \mathrm{Vol}(A)$, which means that f is integrable on A with $\int_A 1 = \int_A f = \mathrm{Vol}(A)$, as required.

7.23 Theorem: *(Linearity) Let $A \subseteq \mathbb{R}^n$ be a Jordan region and let $f, g : A \to \mathbb{R}$ be integrable. Then $f + g$ is integrable, and cf is integrable for every $c \in \mathbb{R}$, and we have*

$$\int_A (f + g) = \int_A f + \int_A g \quad \text{and} \quad \int_A cf = c \int_A f.$$

Proof: The proof is left as an exercise.

7.24 Theorem: *(Decomposition) Let A and B be Jordan regions in $\mathbb{R}^n$ with $\mathrm{Vol}(A \cap B) = 0$, and let $f : A \cup B \to \mathbb{R}$ be bounded. Let $g : A \to \mathbb{R}$ be the restrictions of f to A and let $h : B \to \mathbb{R}$ be the restriction of f to B. Then f is integrable on $A \cup B$ if and only if g is integrable on A and h is integrable on B and, in this case, we have*

$$\int_{A \cup B} f = \int_A g + \int_B h.$$

Proof: The proof is left as an exercise.

7.25 Theorem: *(Comparison) Let A be a Jordan region in $\mathbb{R}^n$ and let $f, g : A \to \mathbb{R}$ be integrable. If $f(x) \leq g(x)$ for all $x \in A$ then $\int_A f \leq \int_A d$.*

Proof: The proof is left as an exercise.

7.26 Theorem: *(Absolute Value) Let A be a Jordan region in $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be integrable. Then the function $|f|$ is integrable and $\left| \int_A f \right| \leq \int_A |f|$.*

Proof: The proof is left as an exercise.

7.27 Theorem: *(Fubini's Theorem for a Rectangle in $\mathbb{R}^2$) Let $R=[a,b]\times[c,d]\subseteq\mathbb{R}^2$, and let $f:R\subseteq\mathbb{R}^2\to\mathbb{R}$ be bounded. For each $x\in[a,b]$ define $g_x:[c,d]\to\mathbb{R}$ by $g_x(y)=f(x,y)$, and for each $y\in[c,d]$, define $h_y:[a,b]\to\mathbb{R}$ by $h_y(x)=f(x,y)$. Suppose that f is integrable on R, g_x is integrable on $[c,d]$ for every $c\in[a,b]$, and h_y is integrable on $[a,b]$ for every $y\in[c,d]$. Then*

$$\iint_R f(x,y)\,dA=\int_{x=a}^b\left(\int_{y=c}^d f(x,y)\;dy\right)dx=\int_{y=c}^d\left(\int_{x=a}^b f(x,y)\;dx\right)dy.$$

Proof: Since g_x and h_y are integrable, we can define $G:[a,b]\to\mathbb{R}$ and $H:[c,d]\to\mathbb{R}$ by

$$G(x)=\int_{y=c}^d g_x(y)\,dy=\int_{y=c}^d f(x,y)\,dy\quad\text{ and }\quad H(y)=\int_{x=a}^b h_y(x)\,dx=\int_{x=a}^b f(x,y)\,dx.$$

Let $\epsilon>0$ and choose a partition Z of R such that $U(f)\le U(f,Z)<U(f)+\epsilon$. Say $Z_1=X=\{x_0,x_1,\cdots,x_n\}$ with $a=x_0<x_1<\cdots<x_n=b$ and $Z_2=Y=\{y_0,y_1,\cdots,y_m\}$ with $c=y_0<y_1<\cdots<y_m=d$. For all indices i,j, let $R_{i,j}=[x_{i-1},x_i]\times[y_{j-1},y_j]$ and let $M_{i,j}=\sup\big\{f(x,y)\,\big|\,(x,y)\in R_{i,j}\big\}$ so that $U(f,Z)=\sum\limits_{i=1}^n\sum\limits_{j=1}^m M_{i,j}(x_i-x_{i-1})(y_j-y_{j-1})$. Note that

$$G(x)=\int_{y=c}^d f(x,y)\,dy=\sum_{j=1}^m G_j(x)\;\text{ where }\;G_j(x)=\int_{y=y_{j-1}}^{y_j} f(x,y)\,dy$$

and note that when $x\in[x_{i-1},x_i]$ we have $G_j(x)\le M_{i,j}(y_{j-1}-y_j)$. Also note that for any bounded maps $\phi,\psi:[a,b]\to\mathbb{R}$ we have $U\big((\phi+\psi),X\big)\le U(\phi,X)+U(\psi,X)$ because $\sup\big\{\phi(x)+\psi(x)\,\big|\,x\in[x_{i-1},x_i]\big\}\le\sup\big\{\phi(x)\,\big|\,x\in[x_{i-1},x_i]\big\}+\sup\big\{\psi(x)\,\big|\,x\in[x_{i-1},x_i]\big\}$. Thus we have

$$\begin{aligned}U(G,X)&=U\big(\sum_{j=1}^m G_j\,,\,X\big)\le\sum_{j=1}^m U(G_j,X)\\&=\sum_{j=1}^m\sum_{i=1}^n\sup\big\{G_j(x)\,\big|\,x\in[x_{i-1},x_i]\big\}\,(x_i-x_{i-1})\\&\le\sum_{j=1}^m\sum_{i=1}^n M_{i,j}(y_j-y_{j-1})(x_i-x_{i-1})=U(f,Z)<U(f)+\epsilon.\end{aligned}$$

Since $U(G)\le U(G,X)<U(f)+\epsilon$ for all $\epsilon>0$, it follows that $U(G)\le U(f)$. A similar argument shows that $L(G)\ge L(f)$, so we have

$$L(f)\le L(G)\le U(G)\le U(f).$$

Since f is integrable so that $L(f)=U(f)$, it follows that $L(f)=L(G)=U(G)=U(f)$ so that G is integrable on $[a,b]$ with $\displaystyle\int_{[a,b]}G=\iint_R f$, that is

$$\iint_R f(x,y)\,dA=\int_{x=a}^b G(x)\,dx=\int_{x=a}^b\left(\int_{y=c}^d f(x,y)\,dy\right)dx.$$

Similarly, $L(f)=L(H)=U(H)=U(f)$ so that

$$\iint_R f(x,y)\,dA=\int_{y=c}^d H(y)\,dy=\int_{x=a}^b\left(\int_{y=c}^d f(x,y)\,dx\right)dy.$$

7.28 Definition: For $\ell \in \{1,2,\cdots,n\}$, the ℓ^{th} **projection map** $p_\ell : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is given by

$$p_\ell(x_1,x_2,\cdots,x_{\ell-1},y,x_\ell,x_{\ell+1},\cdots,x_{n-1}) = (x_1,x_2,\cdots,x_{n-1}).$$

7.29 Theorem: *(Fubini's Theorem for a Rectangle in $\mathbb{R}^n$). Fix $\ell \in \{1,2,\cdots,n\}$. Let $R = [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n] \subseteq \mathbb{R}^n$ and let $S = p_\ell(R) \subseteq \mathbb{R}^{n-1}$. Let $f : R \to \mathbb{R}$ be integrable on R. For each $x \in S$, define $g_x : [a_\ell, b_\ell] \to \mathbb{R}$ by*

$$g_x(y) = f(x_1,\cdots,x_{\ell-1},y,x_\ell,\cdots,x_{n-1})$$

so that $p_\ell(g_x(y)) = x$. Suppose that g_x is integrable on $[a_\ell,b_\ell]$ for every $x \in S$. Define $G : S \to \mathbb{R}$ by

$$G(x) = \int_{y=a_\ell}^{b_\ell} g_x(y)\,dy.$$

Then G is integrable on $[a_\ell,b_\ell]$ and we have

$$\int_R f = \int_S G = \int_S \left(\int_{y=a_\ell}^{b_\ell} g_x(y)\,dy\right) dV = \int_S \left(\int_{y=a_\ell}^{b_\ell} g_x(y)\,dy\right) dx_1 dx_2\cdots dx_{n-1}.$$

Proof: For convenience of notation, we give the proof in the case that $\ell = n$, so we have $S = [a_1,b_1]\times\cdots\times[a_{n-1},b_{n-1}]$, $g_x(y) = f(x,y)$ and $G(x) = \int_{y=a_n}^{b_n} f(x,y)\,dy$, with $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$. Let $\epsilon > 0$. Choose a partition Z of R with $U(f) \leq U(f,Z) < U(f)+\epsilon$. The first $n-1$ components $Z_1, Z_2, \cdots, Z_{n-1}$ of Z determine a partition X of S into sub-rectangles S_k with $k \in K = K(X)$, and the last component of Z gives a partition $Y = Z_n = \{y_0,y_1,\cdots,y_m\}$ of $[a_n,b_n]$, and then Z partitions R into the sub-rectangles $R_{k,j} = S_k \times [y_{j-1},y_j]$ with $|R_{k,j}| = |S_k|(y_j - y_{j-1})$. Let $M_{k,j} = \sup\left\{f(x,y)\,\middle|\,(x,y)\in R_{k,j}\right\}$ so that $U(f,Z) = \sum\limits_{k\in K}\sum\limits_{j=1}^m M_{k,j}|S_k|(y_j - y_{j-1})$.
Note that

$$G(x) = \int_{y=c}^{d} f(x,y)\,dy = \sum_{j=1}^m G_j(x) \quad\text{where}\quad G_j(x) = \int_{y=y_{j-1}}^{y_j} f(x,y)\,dy$$

and note that when $(x,y)\in R_{k,j}$ we have $f(x,y) \leq M_{k,j}$ so $G_j(x) \leq M_{k,j}(y_{j-1}-y_j)$. Also note that for any bounded maps $p,q : S \to \mathbb{R}$ we have $U\big((p+q),X\big) \leq U(p,X) + U(q,X)$ because $\sup\left\{p(x)+q(x)\,\middle|\,x\in S_k\right\} \leq \sup\left\{p(x)\,\middle|\,x\in S_k\right\} + \sup\left\{q(x)\,\middle|\,x\in S_k\right\}$. Thus we have

$$\begin{aligned} U(G,X) &= U\big(\sum_{j=1}^m G_j\,,\,X\big) \leq \sum_{j=1}^m U(G_j,X) = \sum_{j=1}^m\sum_{k\in K} \sup\left\{G_j(x)\,\middle|\,x\in S_k\right\}|S_k| \\ &\leq \sum_{j=1}^m\sum_{k\in K} M_{k,j}(y_j - y_{j-1})\,|S_k| = U(f,Z) < U(f)+\epsilon. \end{aligned}$$

Since $U(G) \leq U(G,X) < U(f)+\epsilon$ for all $\epsilon > 0$, it follows that $U(G) \leq U(f)$. A similar argument shows that $L(G) \geq L(f)$, so we have

$$L(f) \leq L(G) \leq U(G) \leq U(f).$$

Since f is integrable so that $L(f) = U(f)$, it follows that $L(f) = L(G) = U(G) = U(f)$ so that G is integrable on S and $\int_S G = \int_R f$, that is

$$\int_R f = \int_S G = \int_S \left(\int_{y=a_n}^{b_n} f(x,y)\,dy\right) dx_1 dx_2 \cdots dx_{n-1}.$$

7.30 Theorem: *(Iterated Integration) Fix $\ell \in \{1, 2, \cdots, n\}$. Let $B \subseteq \mathbb{R}^{n-1}$ be a closed Jordan region. Let $g, h : B \to \mathbb{R}$ be continuous with $g(x) \leq h(x)$ for all $x \in B$. Let*

$$A = \left\{(x_1, x_2, \cdots, x_{\ell-1}, y, x_\ell, \cdots, x_{n-1}) \in \mathbb{R}^n \,\middle|\, x \in B\,,\ g(x) \leq y \leq h(x)\right\}.$$

Then

(1) A is a Jordan region in $\mathbb{R}^n$, and
(2) when $f : A \to \mathbb{R}$ is continuous, we have

$$\int_A f = \int_B \left(\int_{y=a_\ell}^{b_\ell} f(x_1, \cdots, x_{\ell-1}, y, x_\ell, \cdots, x_{n-1})\, dy \right) dx_1 dx_2 \cdots dx_{n-1}$$

Proof: For notational convenience, we give a proof in the case that $\ell = n$, so we have

$$A = \left\{(x, y) \,\middle|\, x \in B\,,\ g(x) \leq y \leq h(x)\right\}.$$

Verify, as an exercise that $\partial A = C \cup G \cup H$ where

$$C = \left\{(x, y) \,\middle|\, x \in \partial B\,,\ g(x) \leq y \leq h(x)\right\},$$
$$G = \left\{(x, y) \,\middle|\, x \in B\,,\ y = g(x)\right\}, \text{ and}$$
$$H = \left\{(x, y) \,\middle|\, x \in B\,,\ y = h(x)\right\}.$$

Choose a rectangle S in $\mathbb{R}^{n-1}$ which contains B. Note that B is compact and g and h are continuous, hence bounded, so we can choose an interval $[a, b]$ which contains the range of both g and h, so that the rectangle $R = S \times [a, b]$ contains A.

We claim that $U(C) = 0$. Let $\epsilon > 0$. Since B is Jordan measurable we can choose a partition X for S, into sub rectangles S_k with $k \in K$, such that $U(\partial B, X) \leq \frac{\epsilon}{b-a}$. Let Z be the partition of R into sub-rectangles $R_k = S_k \times [a, b]$. Note that for each $k \in K$, we have $R_k \cap C \neq \emptyset \iff S_k \cap \partial B \neq \emptyset$, and hence

$$U(C) \leq U(C, Z) = \sum_{R_k \cap C \neq \emptyset} |R_k| = \sum_{S_k \cap \partial B \neq \emptyset} |S_k|(b-a) = U(\partial B, X)\,(b-a) \leq \epsilon.$$

Since $U(C) \leq \epsilon$ for all $\epsilon > 0$, it follows that $U(C) = 0$, as claimed.

We claim that $U(G) = U(H) = 0$. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ so that $\frac{b-a}{m} \leq \frac{\epsilon}{2(U(B)+1)}$ and let $Y = \{y_0, y_1, \cdots, y_m\}$ be the partition of $[a, b]$ into m equal-sized subintervals, each of size $\frac{b-a}{m}$. Since B is compact and g is continuous, hence uniformly continuous, we can choose $\delta > 0$ so that when $x_1, x_2 \in B$ with $|x_1 - x_2| < \delta$, we have $|g(x_1) - g(x_2)| < \frac{b-a}{2m}$. Choose a partition X of S into sub-rectangles S_k with $k \in K$, so that firstly, we have $U(B, X) \leq U(B) + 1$, and secondly, for each k we have $|x_1 - x_2| < \delta$ for all $x_1, x_2 \in S_k$. Let Z be the partition of R determined by X and Y, that is the partition into the sub-rectangles $R_{k,j} = S_k \times [y_{j-1}, y_j]$. Note that when $R_{k,j} \cap G \neq \emptyset$ we have $S_k \cap B \neq \emptyset$, and note that for each k there are at most 2 values of j for which $R_{k,j} \cap G \neq \emptyset$ because, if we had $(x_i, g(x_i)) \in G \cap R_{k,j_i}$ with $j_1 < j_2 < j_3$ then we would have $x_1, x_3 \in B$ with $g(x_3) - g(x_1) \geq \frac{b-a}{m}$. Thus

$$U(G) \leq U(G, Z) = \sum_{R_{k,j} \cap G \neq \emptyset} |S_k|\, \tfrac{b-a}{m} \leq 2 \cdot \sum_{S_k \cap B \neq \emptyset} |S_k|\, \tfrac{b-a}{m} = 2\, U(B, X)\, \tfrac{b-a}{m} \leq \epsilon.$$

Since $U(G) \leq \epsilon$ for all $\epsilon > 0$, we have $U(B) = 0$. The same argument shows that $U(H) = 0$.

Finally, we note that since $\partial A = C \cup G \cup H$, we have $U(\partial A) \leq \partial C \cup \partial G \cup \partial H = 0$ (by Theorem 7.9), and hence A is Jordan measurable. This completes the proof of Part 1.

To prove Part 2, note that by Definition 7.17 (the definition of the integral), when we extend the domain of a function from a Jordan region to a containing rectangle, by defining the function to be zero outside the Jordan region, the original function is integrable if and only if the extended function is integrable, and they have the same integral. Extend the map $f : A \to \mathbb{R}$ by zero to obtain the map $f : R \to \mathbb{R}$ with $f(x, y) = 0$ when $(x, y) \notin A$. By the definition of the integral, this extended map f is integrable on R with $\int_R f = \int_A f$. By Fubini's Theorem, we have $\int_A f = \int_R f = \int_S G$ where $G(x) = \int_{y=a}^{b} f(x, y)\, dy$, which is integrable on S. When $x \in B$ we have $f(x, y) = 0$ unless $g(x) \leq y \leq h(x)$, and so $G(x) = \int_{y=a}^{b} f(x, y)\, dy = \int_{y=g(x)}^{h(x)} f(x, y)\, dy$. When $x \notin B$ we have $f(x, y) = 0$ for all y so that $G(x) = 0$. By the definition of the integral again, since $G(x) = 0$ whenever $x \notin B$ we have $\int_S G = \int_B G$, and so

$$\int_A f = \int_R f = \int_S G = \int_B G = \int_B \left(\int_{y=g(x)}^{h(x)} f(x, y)\, dy \right) dx_1 dx_2 \cdots dx_{n-1}.$$

7.31 Theorem: *(Local Change of Variables). Let $U \subseteq \mathbb{R}^n$ be open and let $g : U \to \mathbb{R}^n$ be $\mathcal{C}^1$ with $\det Dg \neq 0$ on U. Then for every $a \in U$ there exists an open set W with $a \in W \subseteq U$ such that $g(W)$ is open and $g : W \to g(W)$ is bijective and its inverse is $\mathcal{C}^1$, and such that for every Jordan region A with $\overline{A} \subseteq W$ and for every continuous function $f : g(A) \to \mathbb{R}$, we have*

$$\int_{g(A)} f = \int_A (f \circ g) \big| \det Dg \big|.$$

Proof: We begin by noting that given $a \in U$, using the Inverse Function Theorem we can choose an open set W with $a \in W \subseteq U$ such that $g(W)$ is open and $g : W \to g(W)$ is bijective and its inverse is $\mathcal{C}^1$. Later in the proof we shrink W to make the theorem hold.

We claim that if $|R| = \int_{g^{-1}(R)} \big| \det Dg \big|$ for every rectangle R in $g(W)$, then we have $\int_{g(A)} f = \int_A (f \circ g) \big| \det Dg \big|$ for every Jordan measurable set A with $\overline{A} \subseteq U$ and every continuous function $f : g(A) \to \mathbb{R}$. Suppose that $|R| = \int_{g^{-1}(A)} \big| \det Dg \big|$ for every rectangle R in $g(W)$, let A be a Jordan region with $\overline{A} \subseteq U$ and let $f : g(A) \to \mathbb{R}$ be continuous. Note that the functions $f^+ = \frac{|f|+f}{2}$ and $f^- = \frac{|f|-f}{2}$ are both continuous and non-negative with $f = f^+ - f^-$, so it suffices to consider the case that f is non-negative.

Let $\epsilon > 0$. Choose a rectangle R in $\mathbb{R}^n$ with $g(A) \subseteq R$ and choose a partition X of R into sub-rectangles R_k, $k \in K$ such that $U(f, X) \leq \underline{U(f)} + \epsilon$ and such that for all k, if $R_k \cap \overline{g(A)} \neq \emptyset$ then $R_k \subseteq g(W)$ (we can do this since $\overline{g(A)}$ is compact and $g(W)$ is open). Recall that to obtain $U(f, X)$, we first extend f by zero to all of R, and then we let $M_k = \sup \big\{ f(y) \big| y \in R_k \big\}$. Note that when $R_k \cap g(A) = \emptyset$ we have $M_k = 0$, and so we have $U(f, X) = \sum_{R_k \cap \overline{g(A)} \neq \emptyset} M_k |R_k| = \sum_{R_k \cap g(A) \neq \emptyset} M_k |R_k|$ with

$$M_k = \sup\{ f(y) | y \in R_k \} = \sup\{ f(g(x)) | x \in g^{-1}(R_k) \}.$$

Since the set $\{R_k | R_k \cap g(A) \neq \emptyset\}$ is a set of Jordan regions with disjoint interiors which covers $g(A)$, it follows that the set $\{g^{-1}(R_k) | R_k \cap g(A) \neq \emptyset\}$ is a set of Jordan regions with disjoint interiors which covers A. Let $B = \bigcup_{R_k \cap g(A) \neq \emptyset} g^{-1}(R_k)$. We have

$$\begin{aligned}\int_{g(A)} f + \epsilon &\geq U(f, X) = \sum_{R_k \cap g(A) \neq \emptyset} M_k \, |R_k| = \sum_{R_k \cap g(A) \neq \emptyset} M_k \int_{g^{-1}(R_k)} \big| \det Dg \big| \\ &\geq \sum_{R_k \cap g(A) \neq \emptyset} \int_{g^{-1}(R_k)} (f \circ g) \big| \det Dg \big| = \int_B (f \circ g) \big| \det Dg \big| \\ &\geq \int_A (f \circ g) \big| \det Dg \big|.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that $\int_{g(A)} f \geq \int_A (f \circ g) \big| \det Dg \big|$. A similar argument using $L(f, X)$ shows that $\int_{g(A)} f \leq \int_A (f \circ g) \big| \det Dg \big|$. This proves the claim.

We shall now use the claim to prove the theorem by induction on n. When $n = 1$, the theorem holds by the single variable Change of Variables Theorem. Let $n \geq 2$ and suppose, inductively, that the theorem holds in $\mathbb{R}^{n-1}$. Let $a \in U$. Since $\det Dg(a) \neq 0$, by expanding the determinant along the last row, we see that one of the matrices obtained from $Dg(a)$ by removing the n^{th} row and j^{th} column must have non-zero determinant. For notational convenience, suppose that the upper left $(n-1) \times (n-1)$ submatrix of $Dg(a)$ is invertible. Write elements in W as (x, y) with $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$, re-write the given point $a \in W$ as $(a, b) \in W$, and write $g : W \to g(W)$ as $g(x, y) = \big(h(x, y), g_n(x, y)\big)$ with

$$h(x, y) = \big(g_1(x, y), g_2(x, y), \cdots, g_{n-1}(x, y)\big).$$

Define $p : W \to \mathbb{R}^n$ by

$$p(x, y) = \big(h(x, y), y\big)$$

and note that D_p is the matrix obtained from Dg by replacing the last row by $(0, \cdots, 0, 1)$. In particular $\det Dp(a, b)$ is the determinant of the upper left $(n-1) \times (n-1)$ submatrix of $\det Dg(a, b)$, which we are assuming is non-zero. By the Inverse Function Theorem, we can shrink the open set W, if necessary, so that W and $p(W)$ are open with $(a, b) \in W$, and $p : W \to p(W)$ is invertible with p and p^{-1} both $\mathcal{C}^1$. Define $q : p(W) \to \mathbb{R}^n$ by

$$q(u, v) = \big(u, g_n(p^{-1}(u, v))\big)$$

and note that $q(p(x, y)) = g(x, y)$ for all $(x, y) \in W$ so that g is the composite $g = q \circ p$, and $Dp(x, y) = Dq(p(x, y)) Dp(x, y)$ for all $(x, y) \in W$. The sets W, $p(W)$ and $q(p(W)) = g(W)$ are all open, the maps $g : W \to g(W)$, $p : W \to p(W)$ and $q : p(W) \to q(p(W)) = g(W)$ are all bijective, and these maps and their inverses are all $\mathcal{C}^1$.

Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a rectangle in $p(W)$. let $S = [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]$ so that $R = S \times [a_n, b_n]$. For each $y \in [a_n, b_n]$, define $h : S \to \mathbb{R}^{n-1}$ by $h_y(x) = h(x, y)$. By the induction hypothesis, we have $|S| = \int_{{h_y}^{-1}(S)} \big| \det Dh_y \big|$, and so

$$\begin{aligned}|R| = |S|(b_n - a_n) &= \int_{y=a_n}^{b_n} |S| \, dy = \int_{y=a_n}^{b_n} \int_{{h_y}^{-1}(S)} \big| \det Dh_y \big| \\ &= \int_{y=a_n}^{b_n} \int_{{h_y}^{-1}(S)} \big| \det Dp \big| = \int_{p^{-1}(R)} \big| \det Dp \big|.\end{aligned}$$

By the claim proven above, it follows that for every Jordan measurable set A with $\overline{A} \subseteq W$ and for every continuous map $f : p(A) \to \mathbb{R}$ we have

$$\int_{p(A)} f = \int_A (f \circ p) \big| \det Dp \big| \,. \qquad (1)$$

We can give a similar argument for the function q. Let $R = S \times I$ with $I = [a_n, b_n]$ be a rectangle in $q(p(W)) = g(W)$. For each $u \in S$ let $k_u : I \to \mathbb{R}$ be given by $k_u(v) = k(u,v) = g_n(p^{-1}(u,v))$. By the single variable Change of Variables Theorem, we have $|I| = \int_{k_u{}^{-1}(I)} \big| \det Dk_u \big|$ and so

$$|R| = |S|\,|I| = \int_S |I| = \int_S \int_{k_u{}^{-1}(I)} \big| \det Dk_u \big| = \int_{k_u{}^{-1}(R)} \big| \det Dk_u \big| = \int_{k_u{}^{-1}(R)} \big| \det Dq \big|.$$

By the claim, it follows that for every Jordan measurable set B with $\overline{B} \subseteq p(W)$ and every continuous map $f : q(B) \to \mathbb{R}$ we have

$$\int_{q(B)} f = \int_B (f \circ q) \big| \det Dq \big|. \qquad (2)$$

Combining (1) and (2), we see that for every Jordan measurable set A with $\overline{A} \subseteq W$ and for every continuous map $f : A \to \mathbb{R}$, letting $b = p(A)$ so that $\overline{B} \subseteq p(W)$, we have

$$\begin{aligned}\int_{g(A)=q(B)} f &= \int_{B=p(A)} (f \circ q) \big| \det Dq \big| = \int_A \Big((f \circ q) \big| \det Dq \big| \circ p \Big) \big| \det Dp \big| \\ &= \int_A ((f \circ q) \circ p) \big| (\det Dq) \circ p \big| \, \big| \det Dp \big| = \int_A (f \circ g) \big| \det Dp \big|.\end{aligned}$$

7.32 Theorem: *(Change of Variables) Let $U \subseteq \mathbb{R}^n$ be open, let $g : U \to \mathbb{R}^n$ be $\mathcal{C}^1$ with $\det Dg \neq 0$ on U, let A be a Jordan region with $\overline{A} \subseteq U$, and let $f : g(A) \to \mathbb{R}$ be continuous. Then*

$$\int_{g(A)} f = \int_A (f \circ g) \big| \det Dg \big|.$$

Proof: I may include a proof later.