PMATH 333 Exercises for Appendix 2 (Chapters 1, 2 and 9, and the Appendix)

These optional exercises deal with the material discussed in the first two lectures, and with Cardinality. We discussed a formal symbolic language (called the language of first order set theory), and we discussed the ZFC axioms (which are the rules we allow ourselves to use to construct mathematical sets). If you wish to work on these optional exercises, read Chapters 1, 2 and 9, and the appendix from Appendix 2 (the Algebra Lecture Notes).

1: Recall that a formula in first-order set theory only uses symbols from the following symbol set:

$$\neg\,,\ \wedge\,,\ \vee\,,\ \to\ ,\ \leftrightarrow\ ,\ (\,,\,)\,,\ =\,,\ \in\,,\ \exists\,,\ \forall$$

along with variable symbols including $x, y, z, u, v, w, \cdots$.

(a) Express the statement $\{\emptyset, \{u\}\} \in w$ as a formula in first-order set theory.

(b) Recall that $(x, y) = \big\{\{x\}, \{x, y\}\big\}$. Express the statement "w is a set of ordered pairs" as a formula in first-order set theory.

(c) Express the statement "for every $u \in w$ there exists $x \in u$ such that $u \setminus \{x\} \in w$" as a formula in first order set theory. Also, determine whether there exists such a set w which is not empty.

2: Recall that $\mathbb{N} = \{0, 1, 2, \cdots\}$ is a set where $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$ and in general $x + 1 = x \cup \{x\}$.

(a) Show that if u is a set then the collection $w = \big\{x \cup \{x\}\,\big|\, x \in u\big\}$ is a set.

(b) Show that the collection $w = \big\{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \cdots\big\}$ is a set.

(c) Show that the collection $w = \big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \cdots\big\}$ is a set.

3: In some books on set theory, the list of ZFC axioms includes an additional axiom called the Axiom of Regularity, which states that every nonempty set u contains an element v such that $u \cap v = \emptyset$. Assuming the Axiom of Regularity (along with the other ZFC axioms), prove each of the following statements.

(a) There does not exist a set u such that $u \in u$.

(b) There do not exist sets u and v such that $u \in v$ and $v \in u$.

(c) For all sets u and v, if $u \cup \{u\} = v \cup \{v\}$ then $u = v$.

(d) For all sets u, v, x and y, if $\big\{u, \{u, v\}\big\} = \big\{x, \{x, y\}\big\}$ then $u = x$ and $v = y$.

4: In this problem, and in the following problem, you may use any known properties of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.

(a) Let X and Y be nonempty sets and let $f : X \to Y$. Prove that f is injective if and only if we have $f(A \cap B) = f(A) \cap f(B)$ for all subsets $A, B \subseteq X$.

(b) Show that $|\mathbb{R}| = \big|[0, 1)\big|$ without using the Cantor-Schröder-Bernstein Theorem.

5: (a) Show that the cardinality of the set of all finite subsets of $\mathbb{N}$ is equal to $\aleph_0$.

(b) Show that the cardinality of the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is equal to $2^{\aleph_0}$.