PMATH 333, Exercises for Chapter 2

1: (a) Let $x_k = \frac{2k+1}{k-1}$ for $k \geq 2$. Use the definition of the limit to show that $\lim\limits_{k\to\infty} x_k = 2$ in $\mathbb{R}$.

(b) Let $x_1 = \frac{7}{2}$ and for $k \geq 1$ let $x_{k+1} = \frac{6}{5-a_k}$. Find $\lim\limits_{k\to\infty} x_k$ if it exists in $\mathbb{R}$ (with proof).

(c) Let $(x_k)_{k\geq p}$ and $(y_k)_{k\geq p}$ be sequences in $\mathbb{R}$ with $\lim\limits_{k\to\infty} x_k = c$ where $0 < c \in \mathbb{R}$, and $\lim\limits_{k\to\infty} y_k = \infty$. Use the definition of the limit to show that $\lim\limits_{k\to\infty} \frac{x_k}{y_k} = 0$.

2: (a) Find a divergent sequence $(x_k)_{k\geq 0}$ in $\mathbb{R}$ with $\left|x_k - x_{k-1}\right| \leq \frac{1}{k}$ for all $k \geq 1$.

(b) Let $(x_k)_{k\geq 0}$ be a sequence in $\mathbb{R}$ with $\left|x_k - x_{k-1}\right| \leq \frac{1}{k^2}$ for all $k \geq 1$. Show that (x_k) converges in $\mathbb{R}$.

3: For a sequence $(x_k)_{k\geq p}$ in $\mathbb{R}$ and for $a \in \mathbb{R}$ we say a is a **limiting value** of $(x_k)_{k\geq p}$ when

$$\forall \epsilon > 0 \ \forall m \in \mathbb{Z}_{\geq p} \ \exists k \in \mathbb{Z}_{\geq p} \ (k \geq m \text{ and } |x_k - a| \leq \epsilon).$$

We denote the set of limiting values of $(x_k)_{k\geq p}$ by $\mathrm{Lim}\big((x_k)_{k\geq p}\big)$.

(a) Determine whether, for every sequence $(x_k)_{k\geq p}$ in $\mathbb{R}$, we have $\lim\limits_{k\to\infty} x_k = a \Longrightarrow \mathrm{Lim}\big((x_k)_{k\geq p}\big) = \{a\}$.

(b) Determine whether, for every sequence $(x_k)_{k\geq p}$ in $\mathbb{R}$ we have $\mathrm{Lim}\big((x_k)_{k\geq p}\big) = \{a\} \Longrightarrow \lim\limits_{k\to\infty} x_k = a$.

(c) Determine whether there exists a sequence $(x_k)_{k\geq p}$ in $\mathbb{R}$ with $\mathrm{Lim}\big((x_k)_{k\geq p}\big) = \mathbb{R}$.

4: In this problem, we explore the rate at which the approximations found using Newton's Method approach a square root of a positive real number. Let $a \geq 0$. To approximate $\sqrt{a}$, let $x_1 \geq \sqrt{a}$ and for $k \geq 1$ let $x_{k+1} = \frac{1}{2}\big(x_k + \frac{a}{x_k}\big)$. For $k \geq 1$ let $\epsilon_k = x_k - \sqrt{a}$.

(a) Show that (x_k) is decreasing with $x_k \to \sqrt{a}$.

(b) Show that for all $k \geq 1$ we have $\epsilon_{k+1} = \dfrac{{\epsilon_k}^2}{2\,x_k}$ and that $\dfrac{\epsilon_{k+1}}{2\sqrt{a}} \leq \left(\dfrac{\epsilon_1}{2\sqrt{a}}\right)^{2^k}$.

(c) Show that when $a = 3$ and $x_1 = 2$ we have $\epsilon_6 \leq 4 \cdot 10^{-32}$.

5: Solve the following problems using the definition of the limit and the definition of the derivative as a limit.

(a) Let $f(x) = \frac{1}{x^2-1}$ for $x \neq \pm 1$. Show that $\lim\limits_{x\to 2} f(x) = \frac{1}{3}$.

(b) Let $g(x) = \sqrt{5 - x^2}$ for $|x| \leq \sqrt{5}$. Show that $g'(2) = -2$.

(c) Let $h(x) = \frac{1}{x}$ for $x \neq 0$. Show that $h'(x) = -\frac{1}{x^2}$ for all $x \neq 0$.

6: Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{, if } x \neq 0\,, \\ 0 & \text{, if } x = 0\,, \end{cases}$ and let $g(x) = \begin{cases} 0 & \text{, if } x \notin \mathbb{Q}\,, \\ \frac{1}{b} & \text{, if } x = \frac{a}{b} \text{ with } a \in \mathbb{Z},\, b \in \mathbb{Z}^+ \text{ and } \gcd(a,b) = 1\,. \end{cases}$.

(a) Show that f is differentiable at $x = 0$.

(b) Determine where g is continuous.

(c) Determine where g is differentiable.

7: (a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \cos\big(\pi x^2\big)$. Show that f is not uniformly continuous in $\mathbb{R}$.

(b) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \begin{cases} e^{-1/x^2} & \text{, if } x \neq 0\,, \\ 0 & \text{, if } x = 0\,. \end{cases}$ Use induction to show that $0 = g(0) = g'(0) = g''(0) = \cdots$.

(c) Find a function $h : \mathbb{R} \to \mathbb{R}$ which is differentiable in $\mathbb{R}$ with $h'(0) = 1$ such that for all $\delta > 0$ the function h is not increasing in the interval $(-\delta, \delta)$.

8: (a) Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$. Let $m > 0$ and suppose that $f'(x) \geq m$ for all $x \in [a,b]$. Show that $f(b) \geq f(a) + m(b-a)$.

(b) Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$. Let $m \in \mathbb{R}$ and suppose that $f'(a) < m < f'(b)$. Show that there exists $c \in (a,b)$ such that $f'(c) = m$. (Hint: consider the function $g(x) = f(x) - mx$).

(c) Let $f, g : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ with $f'(x)g(x) = f(x)g'(x)$ for all $x \in [a,b]$. Suppose that $f(a) = f(b) = 0$, $f(x) \neq 0$ for all $x \in (a,b)$, and $g(a) \neq 0$. Show that there exists $c \in (a,b)$ such that $g(c) = 0$.

9: In this problem we explore a uniqueness theorem for differential equations.

(a) Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ with $f(a) = 0$. Suppose that there exists a constant $c > 0$ such that

$$|f'(x)| \le c|f(x)|$$

for all $x \in [a, b]$. Show that $f(x) = 0$ for all $x \in [a, b]$.

(b) Let $A = \big\{(x, y) \,|\, x \in [a, b] \text{ and } y \in [r, s]\big\}$ and let $F : A \to \mathbb{R}$. Suppose there exists a constant $c > 0$ such that

$$\big|F(x, y_1) - F(x, y_2)\big| \le c|y_1 - y_2|$$

for all $x \in [a, b]$ and $y_1, y_2 \in [r, s]$. Show that for each $p \in [r, s]$ there exists at most one function $f : [a, b] \to [r, s]$ with $f(a) = p$ such that $f'(x) = F\big(x, f(x)\big)$ for all $x \in [a, b]$.

(c) Find every function $f : [0, 1] \to [0, 1]$ such that $f'(x) = 2\sqrt{f(x)}$ (there is more than one such function).