PMATH 333 Real Analysis, Solutions to the Exercises for Chapter 2

**1:** (a) Let $x_k = \frac{2k+1}{k-1}$ for $k \geq 2$. Use the definition of the limit to show that $\lim\limits_{k\to\infty} x_k = 2$ in $\mathbb{R}$.

Solution: For $k \geq 2$ and $\epsilon > 0$, we have

$$|x_k - 2| = \left|\frac{2k+1}{k-1} - 2\right| = \left|\frac{2k+1-2k+2}{k-1}\right| = \frac{3}{k-1}$$

and

$$\frac{3}{k-1} < \epsilon \iff \frac{k-1}{3} > \frac{1}{\epsilon} \iff k-1 > \frac{3}{\epsilon} \iff k > 1 + \frac{3}{\epsilon}.$$

Let $\epsilon > 0$. Choose $m \in \mathbb{Z}$ with $m > 1 + \frac{3}{\epsilon}$. For $k \in \mathbb{Z}_{\geq 2}$ with $k \geq m$ we have $k \geq m > 1 + \frac{3}{\epsilon}$ and hence, as shown above, $|x_k - 2| = \frac{3}{k-1} < \epsilon$.

(b) Let $x_1 = \frac{7}{2}$ and for $k \geq 1$ let $x_{k+1} = \frac{6}{5-a_k}$. Find $\lim\limits_{k\to\infty} x_k$ if it exists in $\mathbb{R}$ (with proof).

Solution: Suppose for now that $(x_k)_{k\geq 1}$ does converge, and let $a = \lim\limits_{n\to\infty} x_k$. Then we also have $\lim\limits_{k\to\infty} x_{k+1} = a$ and so taking the limit on both sides of the recursion formula $x_{k+1} = \frac{6}{5-a_k}$ gives

$$a = \frac{6}{5-a} \implies 5a - a^2 = 6 \implies a^2 - 5a + 6 = 0 \implies (a-2)(a-3) = 0,$$

and so we must have $a = 2$ or $a = 3$.

We claim that $x_n < x_{n+1} < 2$ for all $n \geq 4$. We have $x_1 = \frac{7}{2}$, $x_2 = 4$, $x_3 = 6$, $x_4 = -6$ and $x_5 = \frac{6}{11}$, so the claim is true when $n = 4$. Let $k \geq 4$ and suppose the claim is true when $n = k$. Then we have

$$\begin{aligned} x_k < x_{k+1} < 2 &\implies -x_k > -x_{k+1} > -2 \implies 5 - x_k > 5 - x_{k+1} > 3 \implies \frac{1}{5-x_k} < \frac{1}{5-x_{k+1}} < \frac{1}{3} \\ &\implies \frac{6}{5-x_k} < \frac{6}{5-x_{k+1}} < 2 \implies x_{k+1} < x_{k+2} < 2, \end{aligned}$$

so the claim is true when $n = k+1$. By induction, the claim is true for all $n \geq 4$. Thus $(x_n)_{n\geq 4}$ is increasing and is bounded above by 2, so $(x_n)$ converges by the Monotone Convergence Theorem and $\lim\limits_{n\to\infty} x_n \leq 2$ by the Comparison Theorem. We showed above that the limit must be 2 or 3, and so we must have $\lim\limits_{n\to\infty} x_n = 2$.

(c) Let $(x_k)_{k\geq p}$ and $(y_k)_{k\geq p}$ be sequences in $\mathbb{R}$ with $\lim\limits_{k\to\infty} x_k = c$ where $0 < c \in \mathbb{R}$, and $\lim\limits_{k\to\infty} y_k = \infty$. Use the definition of the limit to show that $\lim\limits_{k\to\infty} \frac{x_k}{y_k} = 0$.

Solution: Let $\epsilon > 0$. Since $x_k \to c$ we can choose $m_1 \in \mathbb{Z}$ so that $k \geq m_1 \implies |x_k - c| < \frac{c}{2} \implies \frac{c}{2} < x_k < \frac{3c}{2}$. Since $y_k \to \infty$, we can choose $m_2 \in \mathbb{Z}$ so that $k \geq m_2 \implies y_k > \frac{3c}{2\epsilon}$. Let $m = \max\{m_1, m_2\}$. Then for $k \geq m$ we have $x_k < \frac{3c}{2}$ and we have $y_k > \frac{3c}{2\epsilon}$, and so $\frac{x_k}{y_k} < \frac{3c}{2}\Big/\frac{3c}{2\epsilon} = \epsilon$. Thus $\frac{x_k}{y_k} \to 0$, as required.

**2:** (a) Find a divergent sequence $(x_k)_{k\geq 0}$ in $\mathbb{R}$ with $\left|x_k - x_{k-1}\right| \leq \frac{1}{k}$ for all $k \geq 1$.

Solution: Let $x_0 = 0$ and for $k \geq 1$, let $x_k = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$. Note that $|x_k - x_{k-1}| = x_k - x_{k-1} = \frac{1}{k}$ for all $k \geq 1$. Consider the subsequence $\left(x_{2^k}\right)_{k\geq 0} = (x_1, x_2, x_4, x_8, \cdots)$. We have $x_{2^0} = x_1 = 1$. Let $k \geq 0$ and suppose, inductively, that $x_{2^k} \geq 1 + \frac{k}{2}$. Then

$$\begin{aligned} x_{2^{k+1}} &= \left(\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{2^k}\right) + \left(\tfrac{1}{2^k+1} + \tfrac{1}{2^k+2} + \cdots + \tfrac{1}{2^{k+1}}\right) \\ &= x_{2^k} + \left(\tfrac{1}{2^k+1} + \tfrac{1}{2^k+2} + \cdots + \tfrac{1}{2^{k+1}}\right) \geq x_{2^k} + \left(\tfrac{1}{2^{k+1}} + \tfrac{1}{2^{k+1}} + \cdots + \tfrac{1}{2^{k+1}}\right) \\ &= x_{2^k} + 2^k \cdot \tfrac{1}{2^{k+1}} = x_{2^k} + \tfrac{1}{2} \geq 1 + \tfrac{k}{2} + \tfrac{1}{2} = 1 + \tfrac{k+1}{2}. \end{aligned}$$

By induction, we have $x_{2^n} \geq 1 + \frac{n}{2}$ for all $n \geq 0$. Since $(x_k)$ is increasing and $x_{2^n} \geq 1 + \frac{n}{2}$ for all $n \geq 0$, it follows that $x_k \to \infty$. Indeed, given $r \in \mathbb{R}$ we can choose $n$ so that $1 + \frac{n}{2} > r$ and then for $m = 2^n$ we have $k \geq m \Longrightarrow k \geq 2^n \Longrightarrow x_k \geq x_{2^n} \geq 1 + \frac{n}{2} > r$.

(b) Let $(x_k)_{k\geq 0}$ be a sequence in $\mathbb{R}$ with $\left|x_k - x_{k-1}\right| \leq \frac{1}{k^2}$ for all $k \geq 1$. Show that $(x_k)$ converges in $\mathbb{R}$.

Solution: Notice that for all $k \geq 2$ we have $\frac{1}{k^2} \leq \frac{1}{(k-1)\,k} = \frac{1}{k-1} - \frac{1}{k}$. It follows that for $1 \leq k < l$ we have

$$\begin{aligned} |x_k - x_l| &= \left|x_k - x_{k+1} + x_{k+1} - x_{k+2} + x_{k+2} - x_{k+3} + \cdots - x_{l-1} + x_{l-1} - x_l\right| \\ &\leq |x_k - x_{k+1}| + |x_{k+1} - x_{k+2}| + |x_{k+2} - x_{k+3}| + \cdots + |x_{l-1} - x_l| \\ &\leq \tfrac{1}{(k+1)^2} + \tfrac{1}{(k+2)^2} + \tfrac{1}{(k+3)^2} + \cdots + \tfrac{1}{(l-1)^2} + \tfrac{1}{l^2} \\ &\leq \tfrac{1}{k(k+1)} + \tfrac{1}{(k+1)(k+2)} + \tfrac{1}{(k+2)(k+3)} + \cdots + \tfrac{1}{(l-2)(l-1)} + \tfrac{1}{(l-1)\,l} \\ &= \tfrac{1}{k} - \tfrac{1}{k+1} + \tfrac{1}{k+1} - \tfrac{1}{k+2} + \tfrac{1}{k+2} - \tfrac{1}{k+3} + \cdots - \tfrac{1}{l-1} + \tfrac{1}{l-1} - \tfrac{1}{l} \\ &= \tfrac{1}{k} - \tfrac{1}{l} \leq \tfrac{1}{k}. \end{aligned}$$

Let $\epsilon > 0$. Choose $m \in \mathbb{Z}$ with $m > \frac{1}{\epsilon}$. For $k, l \geq m$ say with $k \leq l$, if $k = l$ then $|x_k - x_l| = 0$ and if $k < l$ then, as shown above, $|x_k - x_l| \leq \frac{1}{k} \leq \frac{1}{m} < \epsilon$. Thus $(x_k)$ is a Cauchy sequence, and so it converges by the Cauchy Criterion.

**3:** For a sequence $(x_k)_{k\geq p}$ in $\mathbb{R}$ and for $a\in\mathbb{R}$ we say $a$ is a **limiting value** of $(x_k)_{k\geq p}$ when

$$\forall\,\epsilon>0\ \forall\,m\in\mathbb{Z}_{\geq p}\ \exists\,k\in\mathbb{Z}_{\geq p}\ (k\geq m \text{ and } |x_k-a|\leq\epsilon).$$

We denote the set of limiting values of $(x_k)_{k\geq p}$ by $\text{Lim}\big((x_k)_{k\geq p}\big)$.

(a) Determine whether, for every sequence $(x_k)_{k\geq p}$ in $\mathbb{R}$, we have $\lim\limits_{k\to\infty} x_k=a \Longrightarrow \text{Lim}\big((x_k)_{k\geq p}\big)=\{a\}$.

Solution: This is true. Let $(x_k)_{k\geq p}$ be a sequence in $\mathbb{R}$ with $x_k\to a$. We claim that $\text{Lim}\big((x_k)\big)=\{a\}$. First we show that $\{a\}\subseteq\text{Lim}\big((x_k)\big)$. Let $\epsilon>0$ and let $m\in\mathbb{Z}_{\geq p}$. Since $x_k\to a$ we can choose $m_0\in\mathbb{Z}_{\geq p}$ so that $k\geq m_0\Longrightarrow |x_k-a|<\epsilon$. Let $k=\max\{m,m_0\}$. Then $k\in\mathbb{Z}_{\geq p}$ with $k\geq m$ and $|x_k-a|<\epsilon$. This proves that $a\in\text{Lim}\big((x_k)\big)$, so we have $\{a\}\subseteq\text{Lim}\big((x_k)\big)$.

Conversely, we need to show that $\text{Lim}\big((x_k)\big)\subseteq\{a\}$. Let $b\in\text{Lim}\big((x_k)\big)$. Suppose, for a contradiction, that $b\neq a$. Since $x_k\to a$, we can choose $m\in\mathbb{Z}_{\geq p}$ so that $k\geq m\Longrightarrow |x_k-a|<\frac{|b-a|}{2}$. Since $b\in\text{Lim}\big((x_k)\big)$, we can choose an index $k$ with $k\geq m$ and $|x_k-b|\leq\frac{|b-a|}{2}$. Then we have

$$|b-a|=|b-x_k+x_k-a|\leq|b-x_k|+|x_k-a|<\tfrac{|b-a|}{2}+\tfrac{|b-a|}{2}=|b-a|,$$

which is not possible. Thus we must have $b=a$, and this shows that $\text{Lim}\big((x_k)\big)\subseteq\{a\}$, as required.

(b) Determine whether, for every sequence $(x_k)_{k\geq p}$ in $\mathbb{R}$ we have $\text{Lim}\big((x_k)_{k\geq p}\big)=\{a\}\Longrightarrow\lim\limits_{k\to\infty}x_k=a$.

Solution: This is false. For example, for the sequence $(x_k)_{k\geq 0}$ given by $x_k=a$ when $k$ is even and $x_k=k$ when $k$ is odd, we have $\text{Lim}\big((x_k)\big)=\{a\}$ but $\lim\limits_{k\to\infty}x_k\neq a$, indeed $(x_k)$ diverges.

Here is a proof that $\text{Lim}\big((x_k)\big)=\{a\}$. Given $\epsilon>0$ and given $m\in\mathbb{N}$ we can choose an even number $k\geq m$ and then we have $|x_k-a|=|a-a|=0\leq\epsilon$. This shows that $a\in\text{Lim}\big((x_k)\big)$ so we have $\{a\}\subseteq\text{Lim}\big((x_k)\big)$. Conversely, let $b\in\text{Lim}\big((x_k)\big)$. Suppose, for a contradiction, that $b\neq a$. Let $\epsilon=\frac{|b-a|}{2}$ and let $m=|b-a|+|b|$. Then for $k\geq m$, if $k$ is even then $x_k=a$ so $|x_k-b|=|a-b|=2\epsilon>\epsilon$, and if $k$ is odd then $x_k=k$ so $|x_k-b|=|k-b|\geq k-|b|\geq m-|b|=|b-a|+|b|-|b|=|b-a|=2\epsilon>\epsilon$. But this contradicts the fact that $b\in\text{Lim}\big((x_k)\big)$. Thus we must have $b=a$, and this shows that $\text{Lim}\big((x_k)\big)\subseteq\{a\}$.

Here is a proof that $\lim\limits_{k\to\infty}x_k\neq a$. Suppose, for a contradiction, that $x_k\to a$. Choose $m\in\mathbb{N}$ so that $k\geq m\Longrightarrow|x_k-a|<1$. Then for all $k\geq m$ we have $a-1<x_k<a+1$. But we can choose an odd number $k\in\mathbb{N}$ with $k\geq\max\{m,a+1\}$ to get $k\geq m$ with $x_k=k\geq a+1$, giving the desired contradiction.

(c) Determine whether there exists a sequence $(x_k)_{k\geq p}$ in $\mathbb{R}$ with $\text{Lim}\big((x_k)_{k\geq p}\big)=\mathbb{R}$.

Solution: There does exist such a sequence $(x_k)$. For example, choose a surjective map $f:\mathbb{Z}^+\to\mathbb{Q}$ and let $x_k=f(k)$ for $k\in\mathbb{Z}^+$. We claim that for this sequence $(x_k)_{k\geq 1}$, we have $\text{Lim}\big((x_k)_{k\geq 1}\big)=\mathbb{R}$. Let $a\in\mathbb{R}$. Let $\epsilon>0$ and let $m\in\mathbb{Z}^+$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose distinct rational numbers $q_1,q_2,q_3,\cdots\in\mathbb{Q}$ with $|q_i-a|\leq\epsilon$ for all $i\geq 1$. For each $i\geq 1$, since $f$ is surjective we can choose $k_i\in\mathbb{Z}^+$ with $f(k_i)=q_i$. Note that the numbers $k_i$ are distinct (since the $q_i$ are distinct and $f$ is a function). Since $k_1,k_2,k_3,\cdots$ are distinct, we can choose an index $j$ such that $k_j\geq m$. For $k=k_j$ we have $k\geq m$ and $|x_k-a|=|f(k)-a|=|q_j-a|\leq\epsilon$. This shows that $a\in\text{Lim}\big((x_k)\big)$. Since $a\in\mathbb{R}$ was arbitrary, we have $\text{Lim}\big((x_k)\big)=\mathbb{R}$.

Here is an example of a surjective map $f:\mathbb{Z}^+\to\mathbb{Q}$: Given $n\in\mathbb{Z}^+$, write $n$ (uniquely in the form) $n=2^k(2\ell-1)$ where $k\in\mathbb{N}$ and $\ell\in\mathbb{Z}^+$. Then define $f(n)=\frac{k/2}{\ell}$ if $k$ is even, and $f(n)=-\frac{(k+1)/2}{\ell}$ is $k$ is odd.

**4:** In this problem, we explore the rate at which the approximations found using Newton's Method approach a square root of a positive real number. Let $a \geq 0$. To approximate $\sqrt{a}$, let $x_1 \geq \sqrt{a}$ and for $k \geq 1$ let $x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$. For $k \geq 1$ let $\epsilon_k = x_k - \sqrt{a}$.

(a) Show that $(x_k)$ is decreasing with $x_k \to \sqrt{a}$.

Solution: We are given that $x_1 \geq \sqrt{a}$. Let $k \geq 1$ and suppose, inductively, that $x_k \geq \sqrt{a}$. Then

$$x_{k+1} - \sqrt{a} = \tfrac{1}{2}\left(x_k + \tfrac{a}{x_k}\right) - \sqrt{a} = \tfrac{1}{2\,x_k}\left({x_k}^2 - 2\sqrt{a}\,x_k + a\right) = \tfrac{1}{2\,x_k}(x_k - \sqrt{a})^2 \geq 0$$

and so $x_{k+1} \geq \sqrt{a}$. By induction, it follows that $x_k \geq \sqrt{a}$ for all $k \geq 1$. This shows that the sequence $(x_k)$ is bounded below by $\sqrt{a}$. For all $k \geq 1$, since $x_k \geq \sqrt{a}$ so that ${x_k}^2 \geq a$, we have

$$x_k - x_{k+1} = x_k - \tfrac{1}{2}\left(x_k + \tfrac{a}{x_k}\right) = \tfrac{1}{2}\left(x_k - \tfrac{a}{x_k}\right) = \tfrac{1}{2\,x_k}\left({x_k}^2 - a\right) \geq 0$$

and so $x_k \geq x_{k+1}$. This shows that the sequence $(x_k)$ is decreasing. Since $(x_k)$ is decreasing and bounded below by $\sqrt{a}$, it converges with $\lim\limits_{k\to\infty} x_k = \sup\{x_k\} \geq \sqrt{a}$. Let $u = \lim\limits_{k\to\infty} x_k$. By taking the limit on both sides of the formula $x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$ we obtain $u = \frac{1}{2}\left(u + \frac{a}{u}\right)$, and

$$u = \tfrac{1}{2}\left(u + \tfrac{a}{u}\right) \implies 2u^2 = u^2 + a \implies u^2 = a \implies u = \pm\sqrt{a} \implies u = \sqrt{a}$$

since we know $u \geq \sqrt{a}$. Thus $x_k \to \sqrt{a}$.

(b) Show that for all $k \geq 1$ we have $\epsilon_{k+1} = \dfrac{{\epsilon_k}^2}{2\,x_k}$ and that $\dfrac{\epsilon_{k+1}}{2\sqrt{a}} \leq \left(\dfrac{\epsilon_1}{2\sqrt{a}}\right)^{2^k}$.

Solution: For $k \geq 1$ we have

$$\epsilon_{k+1} = x_{k+1} - \sqrt{a} = \tfrac{1}{2}\left(x_k + \tfrac{a}{x_k}\right) - \sqrt{a} = \frac{{x_k}^2 - 2x_k\sqrt{a} + a}{2\,x_k} = \frac{(x_k - \sqrt{a})^2}{2\,x_k} = \frac{{\epsilon_k}^2}{2\,x_k}.$$

Since $x_k \geq \sqrt{a}$ this gives $\epsilon_{k+1} = \dfrac{{\epsilon_k}^2}{2\,x_k} \leq \dfrac{{\epsilon_k}^2}{2\sqrt{a}}$ so that $\dfrac{\epsilon_{k+1}}{2\sqrt{a}} \leq \left(\dfrac{\epsilon_k}{2\sqrt{a}}\right)^2$. Using this formula repeatedly, we obtain

$$\frac{\epsilon_{k+1}}{2\sqrt{a}} \leq \left(\frac{\epsilon_k}{2\sqrt{a}}\right)^2 \leq \left(\frac{\epsilon_{k-1}}{2\sqrt{a}}\right)^{2^2} \leq \left(\frac{\epsilon_{k-2}}{2\sqrt{a}}\right)^{2^3} \leq \cdots \leq \left(\frac{\epsilon_1}{2\sqrt{a}}\right)^{2^k}.$$

(c) Show that when $a = 3$ and $x_1 = 2$ we have $\epsilon_6 \leq 4 \cdot 10^{-32}$.

Solution: Let $a = 3$ and $x_1 = 2$. Then $\dfrac{\epsilon_1}{2\sqrt{a}} = \dfrac{x_1 - \sqrt{a}}{2\sqrt{a}} = \frac{2-\sqrt{3}}{2\sqrt{3}} = \frac{1}{\sqrt{3}} - \frac{1}{2}$ Note that

$$\tfrac{1}{\sqrt{3}} - \tfrac{1}{2} \leq \tfrac{1}{10} \iff \tfrac{1}{\sqrt{3}} \leq \tfrac{3}{5} \iff 5 \leq 3\sqrt{3} \iff 25 \leq 9 \cdot 3 = 27,$$

which is true, and so we have $\dfrac{\epsilon_1}{2\sqrt{a}} \leq \frac{1}{10}$. Using the formula $\dfrac{\epsilon_{k+1}}{2\sqrt{a}} \leq \left(\dfrac{\epsilon_1}{2\sqrt{a}}\right)^{2^k}$ with $a = 3$ and $k = 5$, gives

$$\frac{\epsilon_6}{2\sqrt{3}} \leq \left(\frac{\epsilon_1}{2\sqrt{3}}\right)^{32} \leq \left(\tfrac{1}{10}\right)^{32} = 10^{-32}$$

and so $\epsilon_6 \leq 2\sqrt{3} \cdot 10^{-32} \leq 4 \cdot 10^{-32}$.

**5:** Solve the following problems using the definition of the limit and the definition of the derivative as a limit.

(a) Let $f(x) = \frac{1}{x^2-1}$ for $x \neq \pm 1$. Show that $\lim_{x\to 2} f(x) = \frac{1}{3}$.

Solution: First we note that for $x \in \mathbb{R}$ with $x \neq \pm 1$ we have

$$\left|\frac{1}{x^2-1} - \frac{1}{3}\right| = \left|\frac{3-(x^2-1)}{3(x^2-1)}\right| = \left|\frac{4-x^2}{3(x^2-1)}\right| = \frac{|x+2|}{3|x^2-1|} \cdot |x-2|.$$

Next note that when $|x-2| < \frac{1}{2}$ we have $\frac{3}{2} < x < \frac{5}{2}$ so that $\frac{7}{2} < (x+2) < \frac{9}{2}$ and we have $\frac{9}{4} < x^2 < \frac{25}{4}$ so that $\frac{5}{4} < (x^2-1) < \frac{21}{4}$, and so we have $\frac{|x+2|}{3|x^2-1|} = \frac{x+2}{3(x^2-1)} < \frac{\frac{9}{2}}{3\cdot\frac{5}{4}} = \frac{6}{5}$.

Let $\epsilon > 0$. Choose $\delta = \min\left\{\frac{1}{2}, \frac{5\epsilon}{6}\right\}$. Let $x \in \mathbb{R}$ with $0 < |x-2| < \delta$. As shown above, since $|x-2| < \frac{1}{2}$ we have $\frac{|x+2|}{3|x^2-1|} < \frac{6}{5}$, and since $|x-2| < \frac{5\epsilon}{6}$ we have

$$\left|\frac{1}{x^2-1} - \frac{1}{3}\right| = \frac{|x+2|}{3|x^2-1|} \cdot |x-2| < \frac{6}{5} \cdot \frac{5\epsilon}{6} = \epsilon.$$

(b) Let $g(x) = \sqrt{5-x^2}$ for $|x| \leq \sqrt{5}$. Show that $g'(2) = -2$.

Solution: First we note that for $x \in \mathbb{R}$ with $|x| \leq \sqrt{5}$ and $x \neq 2$ we have

$$\begin{aligned}\left|\frac{g(x)-g(2)}{x-2} - (-2)\right| &= \left|\frac{\sqrt{5-x^2}-1}{x-2} + 2\right| = \left|\frac{\sqrt{5-x^2}+2x-5}{x-2}\right| = \left|\frac{\sqrt{5-x^2}+(2x-5)}{x-2} \cdot \frac{\sqrt{5-x^2}-(2x-5)}{\sqrt{5-x^2}-(2x-5)}\right| \\ &= \left|\frac{(5-x^2)-(4x^2-20x+25)}{(x-2)(\sqrt{5-x^2}-(2x-5))}\right| = \left|\frac{-5(x-2)^2}{(x-2)(\sqrt{5-x^2}-(2x-5))}\right| = \frac{5}{\sqrt{5-x^2}+(5-2x)} \cdot |x-2|.\end{aligned}$$

Next note that when $|x-2| < \frac{1}{5}$ we have $\frac{9}{5} < x < \frac{11}{5}$ and since $x < \frac{11}{5}$ we have $x^2 < \frac{121}{25}$ so $5-x^2 > \frac{4}{25}$ so that $\sqrt{5-x^2} > \frac{2}{5}$, and we have $2x < \frac{22}{5}$ so that $5-2x > \frac{3}{5}$, and so we have $\frac{5}{\sqrt{5-x^2}+(5-2x)} < \frac{5}{\frac{2}{5}+\frac{3}{5}} = 5$.

Let $\epsilon > 0$. Choose $\delta = \min\left\{\frac{1}{5}, \frac{\epsilon}{5}\right\}$. Then for $0 < |x-2| < \delta$, as shown above, since $|x-2| < \frac{1}{5}$ we have $\frac{5}{\sqrt{5-x^2}+(5-2x)} < 5$ and since $|x-2| < \frac{\epsilon}{5}$ we have

$$\left|\frac{g(x)-g(2)}{x-2} - (-2)\right| = \frac{5}{\sqrt{5-x^2}+(5-2x)} \cdot |x-2| < 5 \cdot \frac{\epsilon}{5} = \epsilon.$$

(c) Let $h(x) = \frac{1}{x}$ for $x \neq 0$. Show that $h'(x) = -\frac{1}{x^2}$ for all $x \neq 0$.

Solution: First note that for $x \neq 0$, $u \neq 0$ and $u \neq x$ we have

$$\left|\frac{h(u)-h(x)}{u-x} - \left(-\frac{1}{x^2}\right)\right| = \left|\frac{\frac{1}{u}-\frac{1}{x}}{u-x} + \frac{1}{x^2}\right| = \left|\frac{x-u}{ux(u-x)} + \frac{1}{x^2}\right| = \left|-\frac{1}{ux} + \frac{1}{x^2}\right| = \left|\frac{u-x}{ux^2}\right| = \frac{1}{|u|\,|x|^2} \cdot |u-x|.$$

Next note that when $|u-x| < \frac{|x|}{2}$ we have $|x| = |(x-u)+u| \leq |x-u|+|u| < \frac{|x|}{2}+|u|$ so that $|u| > |x| - \frac{|x|}{2} = \frac{|x|}{2}$ and hence $\frac{1}{|u|\,|x|^2} < \frac{1}{\frac{|x|}{2}\cdot|x|^2} = \frac{2}{|x|^3}$.

Let $x \in \mathbb{R}$ with $x \neq 0$. Let $\epsilon > 0$. Choose $\delta = \min\left\{\frac{|x|}{2}, \frac{|x|^3\epsilon}{2}\right\}$. Then for $u \in \mathbb{R}$ with $|u-x| < \delta$, as shown above, since $|u-x| < \frac{|x|}{2}$ we have $\frac{1}{|u|\,|x|^2} < \frac{2}{|x|^3}$ and since $|u-x| < \frac{|x|^3\epsilon}{2}$ we have

$$\left|\frac{h(u)-h(x)}{u-x} - \left(-\frac{1}{x^2}\right)\right| = \frac{1}{|u|\,|x|^2} \cdot |u-x| < \frac{2}{|x|^3} \cdot \frac{|x|^3\epsilon}{2} = \epsilon.$$

**6:** Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & , \text{ if } x \neq 0\,, \\ 0 & , \text{ if } x = 0\,, \end{cases}$ and let $g(x) = \begin{cases} 0 & , \text{ if } x \notin \mathbb{Q}\,, \\ \frac{1}{b} & , \text{ if } x = \frac{a}{b} \text{ with } a \in \mathbb{Z},\, b \in \mathbb{Z}^+ \text{ and } \gcd(a,b) = 1\,. \end{cases}$.

(a) Show that $f$ is differentiable at $x = 0$.

Solution: We claim that $f$ is differentiable at 0 with $f'(0) = 0$. Let $\epsilon > 0$. Choose $\delta = \epsilon$. For $x \in \mathbb{R}$ with $0 < |x - 0| < \delta$ we have $0 < |x| < \epsilon$ and so

$$\left|\frac{f(x)-f(0)}{x-0} - 0\right| = \left|\frac{x^2 \sin\frac{1}{x} - 0}{x-0} - 0\right| = \left|x \sin \tfrac{1}{x}\right| = |x|\left|\sin\tfrac{1}{x}\right| \leq |x| \cdot 1 < \epsilon$$

since $|\sin u| \leq 1$ for all $u \in \mathbb{R}$.

(b) Determine where $g$ is continuous.

Solution: We claim that $g$ is continuous at $a \in \mathbb{R}$ if and only if $a \notin \mathbb{Q}$. Suppose first that $a \in \mathbb{Q}$, say $a = \frac{k}{n}$ with $k \in \mathbb{Z}$, $n \in \mathbb{Z}^+$ and $\gcd(k,n) = 1$ so that $g(a) = \frac{1}{n}$. We claim that $g$ is not continuous at $a$ $\big($we need to show that there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x \in \mathbb{R}$ such that $|x - a| < \delta$ and $|g(x) - g(a)| \geq \epsilon\big)$. Choose $\epsilon = \frac{1}{n}$. Let $\delta > 0$. Choose $x \in \mathbb{R}$ with $x \notin \mathbb{Q}$ and $|x - a| < \delta$ $\big($for example, choose $m \in \mathbb{Z}^+$ with $m > \frac{\sqrt{2}}{\delta}$ and then let $x = a + \frac{\sqrt{2}}{m}\big)$. Then we have $g(x) = 0$ and $g(a) = \frac{1}{n}$ and so $|g(x) - g(a)| = \frac{1}{n} = \epsilon$.

Next suppose that $a \notin \mathbb{Q}$ and note that $g(a) = 0$. We claim that $g(x)$ is continuous at $a$. Let $\epsilon > 0$. Choose $n \in \mathbb{Z}^+$ with $\frac{1}{n} < \epsilon$. Let $S$ be the set of all points $x \in [a-1, a+1]$ of the form $x = \frac{k}{m}$ with $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$ with $m \leq n$ (we remark that $S$ is not empty because $\lfloor a \rfloor \in S$). Note that there are only finitely many points in $S$ since for each choice of $m \in \mathbb{Z}^+$ with $m \leq n$ there are only finitely many $k \in \mathbb{Z}$ with $m(a-1) < k < m(a+1)$. Choose $\delta = \min\big\{|x - a|\big| x \in S\big\}$ (we remark that $\delta < 1$ because $\lfloor a \rfloor \in S$). Note that $\delta > 0$ since $a \notin \mathbb{Q}$ so $a \notin S$ and so $|x - a| > 0$ for every $x \in S$. For $0 < |x - a| < \delta$, either $x \notin \mathbb{Q}$ in which case $g(x) = 0$ so that $|g(x) - g(a)| = 0 < \epsilon$, or $x \in \mathbb{Q}$ in which case $x \notin S$ (since $|x - a| \geq \delta$ for all $x \in S$) and so when we write $x = \frac{k}{m}$ with $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$ and $\gcd(k,m) = 1$ we must have $m > n$ and so $|g(x) - f(x)| = \frac{1}{m} < \frac{1}{n} < \epsilon$.

(c) Determine where $g$ is differentiable.

Solution: We claim that $g$ is not differentiable at any point $a \in \mathbb{R}$. When $a \in \mathbb{Q}$ we know from Part (b) that $g$ is not continuous at $a$, and so $g$ is not differentiable at $a$. Suppose that $a \notin \mathbb{Q}$. Suppose, for a contradiction, that $g$ is differentiable at $a$. Take $\epsilon = \frac{1}{2}$ in the definition of differentiability and choose $\delta > 0$ so that for all $x \in \mathbb{R}$, if $0 < |x - a| < \delta$ then $\left|\frac{g(x)-g(a)}{x-a} - g'(a)\right| < \frac{1}{2}$, that is

$$\frac{g(x)-g(a)}{x-a} - \frac{1}{2} < g'(a) < \frac{g(x)-g(a)}{x-a} + \frac{1}{2}.$$

Choose a prime number $p \in \mathbb{Z}^+$ so that $\frac{2}{p} < \delta$ (we can do this because there are infinitely many prime numbers). Let $k = \lfloor ap \rfloor$ (we remark that $k \neq ap$ since $a \notin \mathbb{Q}$). Then we have $ap - 2 < k - 1 < k < ap$ so that $a - \frac{2}{p} < \frac{k-1}{p} < \frac{k}{p} < a$, and we have $ap < k + 1 < k + 2 < ap + 2$ so that $a < \frac{k+1}{p} < \frac{k+2}{p} < a + \frac{2}{p}$. Pick $k_1 \in \{k-1, k\}$ with $p \nmid k_1$ so that $\gcd(k_1, p) = 1$ and let $x_1 = \frac{k_1}{p}$. Pick $k_2 \in \{k+1, k+2\}$ with $p \nmid k_2$ so that $\gcd(k_2, p) = 1$ and let $x_2 = \frac{k_2}{p}$. Then we have $a - \delta < a - \frac{1}{2p} < x_1 < a < x_2 < a + \frac{2}{p} < a + \delta$ and $g(x_1) = g(x_2) = \frac{1}{p}$. It follows that

$$\frac{g(x_1)-g(a)}{x_1-a} < -\frac{1/p}{2/p} = -\frac{1}{2} \quad \text{and} \quad \frac{g(x_2)-g(a)}{x_2,a} > \frac{1/p}{2/p} = \frac{1}{2}$$

and hence that

$$g'(a) < \frac{g(x_1)-g(a)}{x_1-a} + \frac{1}{2} < -\frac{1}{2} + \frac{1}{2} = 0 \quad \text{and} \quad g'(a) > \frac{g(x_2)-g(a)}{x_2-a} - \frac{1}{2} > \frac{1}{2} - \frac{1}{2} > 0$$

which gives us the desired contradiction.

**7:** (a) Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=\cos\left(\pi x^2\right)$. Show that $f$ is not uniformly continuous in $\mathbb{R}$.

Solution: Note that $f(\sqrt{n})=\cos(\pi n)=(-1)^n$ for all $n\in\mathbb{Z}^+$. To show that $f$ is not uniformly continuous in $\mathbb{R}$ we need to show that there exists $\epsilon>0$ such that for all $\delta>0$ there exists $a,x\in\mathbb{R}$ such that $|x-a|<\delta$ and $|f(x)-f(a)|\geq\epsilon$. Choose $\epsilon=1$. Let $\delta>0$. Since $\sqrt{n+1}-\sqrt{n}=\frac{1}{\sqrt{n+1}+\sqrt{n}}\to 0$ as $n\to\infty$, we can choose $n\in\mathbb{Z}^+$ so that $\sqrt{n+1}-\sqrt{n}<\delta$. Then for $a=\sqrt{n}$ and $x=\sqrt{n+1}$ we have $|x-a|<\delta$ but $|f(x)-f(a)|=\left|(-1)^{n+1}-(-1)^n\right|=2>\epsilon$.

(b) Define $g:\mathbb{R}\to\mathbb{R}$ by $g(x)=\begin{cases} e^{-1/x^2} & \text{, if } x\neq 0\,, \\ 0 & \text{, if } x=0\,. \end{cases}$ Use induction to show that $0=g(0)=g'(0)=g''(0)=\cdots$.

Solution: When $x\neq 0$ we have $g'(x)=\frac{2}{x^3}\,e^{-1/x^2}$ and $g''(x)=\left(\frac{4}{x^6}-\frac{6}{x^4}\right)e^{-1/x^2}$. Let $n\geq 1$ and suppose, inductively, that $g^{(n)}(x)=p_n\left(\frac{1}{x}\right)e^{-1/x^2}$ where $p_n(t)$ is a polynomial of degree $3n$. Then

$$g^{(n+1)}(x)=p_n{}'\left(\tfrac{1}{x}\right)\cdot\left(-\tfrac{1}{x^2}\right)e^{-1/x^2}+p_n\left(\tfrac{1}{x}\right)\cdot\left(\tfrac{2}{x^3}\right)\,e^{-1/x^2}=p_{n+1}\left(\tfrac{1}{x}\right)e^{-1/x^2}$$

where $p_{n+1}(t)=2t^3p_n(t)-t^2p_n{}'(t)$, which is a polynomial of degree $3(n+1)$. By Induction, it follows that for $x\neq 0$ we have $g^{(n)}(x)=p_n\left(\frac{1}{x}\right)e^{-1/x^2}$ for all $n\geq 0$, where $p_n(t)$ is the polynomial of degree $3n$ defined recursively by $p_0(x)=1$ and $p_{n+1}(t)=2t^3p_n(t)-t^2p_n{}'(t)$ for $n\geq 0$. From the definition of the derivative we have

$$g'(0)=\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}=\lim_{x\to 0}\frac{e^{-1/x^2}-0}{x-0}=\lim_{x\to 0}\frac{1/x}{e^{1/x^2}}.$$

As $x\to 0$ we have $\frac{1}{x^2}\to\infty$ so $e^{1/x^2}\to\infty$, as $x\to 0^+$ we have $\frac{1}{x}\to+\infty$ and as $x\to 0^-$ we have $\frac{1}{x}\to-\infty$, and so by l'Hôpital's Rule, we have

$$g'(0)=\lim_{x\to 0}\frac{\frac{1}{x}}{e^{1/x^2}}=\lim_{x\to 0}\frac{-\frac{1}{x^2}}{-\frac{2}{x^3}\,e^{1/x^2}}=\lim_{x\to 0}\frac{x}{2\,e^{1/x^2}}=\frac{0}{\infty}=0.$$

Let $n\geq 0$ and suppose, inductively, that $g^{(n)}(0)=0$. Then we have

$$g^{(n)}(x)=\begin{cases} p_n\left(\frac{1}{x}\right)e^{-1/x^2} & \text{if } x\neq 0, \\ 0 & \text{if } x=0. \end{cases}$$

From the definition of the derivative, we have

$$g^{(n+1)}(0)=\lim_{x\to 0}\frac{g^{(n)}(x)-g^{(n)}(0)}{x-0}=\lim_{x\to 0}\frac{p_n\left(\frac{1}{x}\right)\,e^{-1/x^2}}{x}=\lim_{x\to 0}\frac{\frac{1}{x}\,p_n\left(\frac{1}{x}\right)}{e^{1/x^2}}.$$

Note that in order to show that $\displaystyle\lim_{x\to 0}\frac{\frac{1}{x}\,p_n\left(\frac{1}{x}\right)}{e^{1/x^2}}=0$, it suffices to show that $\displaystyle\lim_{x\to 0}\frac{1/x^k}{e^{1/x^2}}=0$ for all $k\geq 0$ (because $\frac{\frac{1}{x}\,p_n\left(\frac{1}{x}\right)}{e^{1/x^2}}$ is equal to a sum of terms of the form $\frac{1/x^k}{e^{1/x^2}}$ ). We already know that this is true when $k=0$ and when $k=1$ (we shall need two base cases). Let $k\geq 0$ and suppose, inductively, that $\displaystyle\lim_{x\to 0}\frac{1/x^k}{e^{1/x^2}}=0$. Then by l'Hôpital's Rule yet again we have

$$\lim_{x\to 0}\frac{\frac{1}{x^{k+2}}}{e^{1/x^2}}=\lim_{x\to 0}\frac{-\frac{k+2}{x^{k+3}}}{-\frac{2}{x^3}\,e^{1/x^2}}=\lim_{x\to 0}\frac{k+2}{2}\cdot\frac{\frac{1}{x^k}}{e^{1/x^2}}=0.$$

By the Strong Induction Principle, it follows that $\displaystyle\lim_{x\to 0}\frac{1/x^k}{e^{1/x^2}}=0$ for all $n\geq 0$, and so $\displaystyle\lim_{x\to 0}\frac{\frac{1}{x}\,p_n\left(\frac{1}{x}\right)}{e^{1/x^2}}=0$ for all $n\geq 0$, hence $g^{(n)}(0)=\displaystyle\lim_{x\to 0}\frac{\frac{1}{x}\,p_n\left(\frac{1}{x}\right)}{e^{1/x^2}}=0$ for all $n\geq 0$, as required.

(c) Find a function $h:\mathbb{R}\to\mathbb{R}$ which is differentiable in $\mathbb{R}$ with $h'(0)=1$ such that for all $\delta>0$ the function $h$ is not increasing in the interval $(-\delta,\delta)$.

Solution: Let $h(x)=x+2f(x)$ where $f(x)$ is the function from Problem 2. By Part (a) of Problem 2 we have $h'(0)=1+0=1$, and for $x\neq 0$ we have $h(x)=x+2x^2\sin\frac{1}{x}$ so that $h'(x)=1+4x\sin\frac{1}{x}-2\cos\frac{1}{x}$. Note that $h'(x)$ is continuous for all $x\neq 0$. Let $\delta>0$. We claim that $h(x)$ is not increasing in the interval $(-\delta,\delta)$. Choose $k\in\mathbb{Z}^+$ so that $\frac{1}{\pi k}<\delta$ and let $a=\frac{1}{2\pi k}$. Since $\sin 2\pi k=0$ and $\cos 2\pi k=1$ we have $h'(a)=1+0-2=-1$. Since $h'(x)$ is continuous for $x>0$ we can choose $\delta_1$ with $0<\delta_1<\frac{1}{2\pi k}$ so that for all $x>0$ with $|x-a|<\delta_1$ we have $|h'(x)-h'(a)|<\frac{1}{2}$ and hence $h'(x)<h'(a)+\frac{1}{2}=-1+\frac{1}{2}=-\frac{1}{2}$. Since $h'(x)<-\frac{1}{2}<0$ for all $x\in(a-\delta_1,a+\delta_1)$, it follows that $h(x)$ is decreasing in the interval $(a-\delta_1,a+\delta_1)$. Since $(a-\delta_1,a+\delta_1)\subseteq\left(0,\frac{1}{\pi k}\right)\subseteq(-\delta,\delta)$, it follows that $h(x)$ is not increasing in the interval $(-\delta,\delta)$.

**8:** (a) Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Let $m > 0$ and suppose that $f'(x) \geq m$ for all $x \in [a, b]$. Show that $f(b) \geq f(a) + m(b - a)$.

Solution: By the Mean Value Theorem, we can choose $x \in [a, b]$ such that $\frac{f(b)-f(a)}{b-a} = f'(x)$ and then, since $f'(x) \geq m$, we have $\frac{f(b)-f(a)}{b-a} \geq m$ so that $f(b) - f(a) \geq m(b - a)$ and hence $f(b) \geq f(a) + m(b - a)$.

(b) Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Let $m \in \mathbb{R}$ and suppose that $f'(a) < m < f'(b)$. Show that there exists $c \in (a, b)$ such that $f'(c) = m$. (Hint: consider the function $g(x) = f(x) - mx$).

Solution: Let $g(x) = f(x) - mx$. Since $f$ is differentiable on $[a, b]$, so is $g$ and we have $g'(x) = f'(x) - m$. Since $g$ is differentiable on $[a, b]$, it follows that $g$ is also continuous on $[a, b]$, and so by the Extreme Value Theorem, $g$ attains its minimum value on $[a, b]$. Choose $c \in [a, b]$ so that $g(c) \leq g(x)$ for all $x \in [a, b]$. Since $f'(a) < m$ we have $g'(a) = f'(a) - m < 0$ and so $g$ does not attain its minimum value at $a$ $\Big($indeed we can choose $\delta > 0$ such that for all $x$ with $a < x < a + \delta$ we have $\left|\frac{g(x)-g(a)}{x-a} - g'(a)\right| < \frac{|g'(a)|}{2}$ so that $\frac{g(x)-g(a)}{x-a} < g'(a) + \frac{|g'(a)|}{2} = g'(a) - \frac{g'(a)}{2} = \frac{g'(a)}{2} < 0$ which implies that $g(x) = g(a) < 0$ so that $g(x) < g(a)\Big)$. Since $f'(b) > m$ we have $g'(b) = f'(b) - m > 0$ and so $g$ does not attain its minimum value at $b$. Since $g$ does not attain its minimum value at $a$ or $b$ we must have $c \in (a, b)$. Since $g$ has a minimum value at $c \in (a, b)$, it follows from Fermat's Theorem that $g'(c) = 0$, and hence $f'(c) = g'(c) + m = m$.

(c) Let $f, g : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ with $f'(x)g(x) = f(x)g'(x)$ for all $x \in [a, b]$. Suppose that $f(a) = f(b) = 0$, $f(x) \neq 0$ for all $x \in (a, b)$, and $g(a) \neq 0$. Show that there exists $c \in (a, b)$ such that $g(c) = 0$.

Solution: Suppose, for a contradiction, that $g'(x) \neq 0$ for all $x \in (a, b)$. Since $g(a) \neq 0$ we have $g(x) \neq 0$ for all $x \in [a, b)$. Since $f$ and $g$ are differentiable with $g(x) \neq 0$ for all $x \in [a, b)$ it follows that the function $h(x) = \frac{f(x)}{g(x)}$ is differentiable with $h'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$ for all $x \in [a, b)$. Since $f'(x)g(x) = f(x)g'(x)$ for all $x \in [a, b]$ it follows that $h'(x) = 0$ for all $x \in [a, b)$. Since $h'(x) = 0$ for all $x \in [a, b)$ it follows that $h$ is constant in $[a, b)$. Since $h$ is constant in $[a, b)$ with $h(a) = \frac{f(a)}{g(a)} = \frac{0}{g(a)} = 0$, it follows that $h(x) = 0$ for all $x \in [a, b)$. This gives the desired contradiction because for all $x \in (a, b)$ we have $f(x) \neq 0$ and $g(x) \neq 0$ so that $h(x) = \frac{f(x)}{g(x)} \neq 0$.

**9:** In this problem we explore a uniqueness theorem for differential equations.

(a) Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ with $f(a) = 0$. Suppose that there exists a constant $c > 0$ such that

$$|f'(x)| \le c|f(x)|$$

for all $x \in [a, b]$. Show that $f(x) = 0$ for all $x \in [a, b]$.

Solution: We wish to show that $f(x) = 0$ for all $x \in [a, b]$. We have $f(a) = 0$. Let $k \in \mathbb{N}$ and suppose, inductively, that $f(x) = 0$ for all $x \in [a, b]$ with $a \le x \le a + \frac{k}{2c}$. We need to show that $f(x) = 0$ for all $x \in [a, b]$ with $a + \frac{k}{2c} < x \le a + \frac{k+1}{2c}$. If $a + \frac{k}{c} \ge b$ then we have $f(x) = 0$ for all $x \in [a, b]$ so there is nothing to prove. Suppose that $a + \frac{k}{2c} < b$. To simplify our notation, write $d = a + \frac{k}{2c} < b$. Then we have $f(x) = 0$ for all $x \in [a, d]$ and we need to show that $f(x) = 0$ for all $x \in [a, b]$ with $d < x \le d + \frac{1}{2c}$.

Let $x \in [a, b]$ with $d < x \le d + \frac{1}{2c}$. Let $\ell = \sup\big\{|f(t)|\big|d \le t \le x\big\}$ and let $m = \sup\big\{|f'(t)|\big|d \le t \le x\big\}$.

We claim that $m \le c\ell$. Suppose, for a contradiction, that $m > c\ell$. Let $\epsilon = m - c\ell$. By the Approximation Property, we can choose $t \in [d, x]$ so that $m - \epsilon < |f'(t)| \le m$. Then we have $m - \epsilon < |f'(t)| \le c|f(t)| \le c\ell$. But then $\epsilon > m - c\ell = \epsilon$ giving the desired contradiction. Thus $m \le c\ell$, as claimed.

Next, we claim that $|f(t)| \le \frac{\ell}{2}$ for all $t \in [d, x]$. We know that $f(d) = 0$, so suppose that $t \in (d, x]$. By the Mean Value Theorem, we can choose $s \in (d, x)$ such that $f'(s) = \frac{f(t)-f(d)}{t-d} = \frac{f(t)}{t-d}$. and then $f(t) = f'(s)(t - d)$. It follows that $|f(t)| = |f'(s)|(t - d) \le m(t - d) \le m(x - d) \le c\ell(x - d) \le c\ell \cdot \frac{1}{2c} = \frac{\ell}{2}$, as claimed.

We claim that $\ell = 0$. Note that since $\ell = \sup\big\{|f(t)\big|t \in [d, x]\big\}$ we have $\ell \ge |f(d)| = 0$. Suppose, for a contradiction, that $\ell > 0$. By the Approximation Property, we can choose $t \in [d, x]$ such that $\frac{\ell}{2} < |f(t)| \le \ell$. But this contradicts the fact that $|f(t)| \le \frac{\ell}{2}$ for all $t \in [d, x]$ (as we just proved)and so $\ell = 0$, as claimed.

Finally note that since $\ell = \sup\big\{|f(t)|\big|t \in [d, x]\big\} = 0$ it follows that $|f(t)| = 0$ for all $t \in [d, x]$, so in particular $f(x) = 0$. Since $x$ was arbitrary, this proves that $f(x) = 0$ for every $x \in [a, b]$ with $d < x \le d + \frac{1}{2c}$, as required.

(b) Let $A = \big\{(x, y)\,|\,x \in [a, b] \text{ and } y \in [r, s]\big\}$ and let $F : A \to \mathbb{R}$. Suppose there exists a constant $c > 0$ such that

$$\big|F(x, y_1) - F(x, y_2)\big| \le c|y_1 - y_2|$$

for all $x \in [a, b]$ and $y_1, y_2 \in [r, s]$. Show that for each $p \in [r, s]$ there exists at most one function $f : [a, b] \to [r, s]$ with $f(a) = p$ such that $f'(x) = F\big(x, f(x)\big)$ for all $x \in [a, b]$.

Solution: Suppose that $f_1, f_2 : [a, b] \to [r, s]$ with $f_1(a) = f_2(a) = p$, $f_1'(t) = F\big(x, f_1(x)\big)$ and $f_2'(x) = F\big(x, f_2(x)\big)$. We must show that $f_1(x) = f_2(x)$ for all $x \in [0, 1]$. Let $f(x) = f_1(x) = f_2(x)$. Then for all $x \in [0, 1]$ we have

$$|f'(x)| = \big|f_1'(x) - f_2'(x)\big| = \Big|F\big(x, f_1(x)\big) - F\big(x, f_2(x)\big)\Big| \le c\big|f_1(x) - f_2(x)\big| = c|f(x)|.$$

By Part (1) it follows that $f'(x) = 0$ for all $x \in [0, 1]$. Since $f'(x) = 0$ for all $x \in [0, 1]$ it follows that $f(x)$ is constant. Since $f(0) = f_1(0) - f_2(0) = p - p = 0$ and $f(x)$ is constant, it follows that $f(x) = 0$ for all $x \in [0, 1]$. Thus for all $x \in [0, 1]$ we have $0 = f(x) = f_1(x) - f_2(x)$ so that $f_1(x) = f_2(x)$, as required.

(c) Find every function $f:[0,1]\to[0,1]$ such that $f'(x)=2\sqrt{f(x)}$ (there is more than one such function).

Solution: Will will show that the required functions are given by

$$f(x)=\begin{cases}0 & \text{if } 0\le x\le c,\\ (x-c)^2 & \text{if } c\le x\le 1.\end{cases}$$

where $c$ is a constant with $0\le c\le 1$. Note that $f$ is increasing with $f(0)=0$ and $f(1)=(1-c)^2\le 1$.

First let us show that for the above functions $f$ we do indeed have $f'(x)=2\sqrt{f(x)}$ for all $x\in[0,1]$. When $0\le x\le c$ we have $\sqrt{f(x)}=\sqrt{0}=0$ and when $c\le x\le 1$ we have $\sqrt{f(x)}=\sqrt{(x-c)^2}=|x-c|=x-c$. On the other hand, when $0\le x<c$ we have $f(x)=0$ so that $f'(x)=0$, and when $c<x\le 1$ we have $f(x)=(x-c)^2$ so that $f'(x)=2(x-c)$, and we have

$$\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c^-}\frac{0-0}{x-c}=0 \text{ and } \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c^+}\frac{(x-c)^2-0}{x-c}=\lim_{x\to c^+}(x-c)=0$$

so that $f'(c)=0$, and so in all cases we have $f'(x)=2\sqrt{f(x)}$, as required. It remains to show that we have found all of the solutions.

Let $f$ be any function $f:[0,1]\to[0,1]$ with $f'(x)=2\sqrt{f(x)}$ for all $x\in[0,1]$. We remark that $f$ is differentiable because $f'(x)$ exists. We also remark that $f$ must be increasing on $[0,1]$ because $f'(x)=\sqrt{f(x)}\ge 0$ for all $x\in[0,1]$.

First we claim that for any nonempty interval $I\subseteq[0,1]$, if $f(x)>0$ for all $x\in I$ then there exists $b\in\mathbb{R}$ such that $f(x)=(x-b)^2$ for all $x\in I$. Let $I$ be any nonempty interval with $I\subseteq[0,1]$ and suppose that $f(x)>0$ for all $x\in I$. Let $g(x)=2\sqrt{f(x)}-2x$ for all $x\in I$. Then $g$ is differentiable in $I$ (since $f$ is differentiable and the function $\sqrt{u}$ is differentiable for $u>0$) with $g'(x)=\frac{f'(x)}{\sqrt{f(x)}}-2=\frac{2\sqrt{f(x)}}{\sqrt{f(x)}}-2=0$. Since $g'(x)=0$ for all $x\in I$ it follows that $g$ is constant in $I$. Choose $a\in I$. Then for all $x\in I$ we have $2\sqrt{f(x)}-2x=g(x)=g(a)=2\sqrt{f(a)}-2a$ and so $f(x)=\big(x+\sqrt{f(a)}-a\big)^2$. Thus we have $f(x)=(x-b)^2$ for all $x\in I$, where $b=a-\sqrt{f(a)}$.

Next we claim that $f(0)=0$. Suppose, for a contradiction, that $f(0)=p>0$. Since $f$ is increasing on $[0,1]$ with $f(0)=p>0$, we have $f(x)\ge f(0)=p>0$ for all $x\in[0,1]$. By the previous paragraph, we can choose $b\in\mathbb{R}$ so that $f(x)=(x-b)^2$ for all $x\in I$. Since $f'(x)=2\sqrt{f(x)}$ we have $f'(0)=2\sqrt{p}$ and since $f(x)=(x-b)^2$ we have $f'(x)=2(x-b)$ and so $f'(0)=-2b$. It follows that $-2b=2\sqrt{p}$ so that $b=-\sqrt{p}$. Thus we must have $f(x)=(x+\sqrt{p})^2$ for all $x\in[0,1]$. In particular, we must have $f(1)=(1+\sqrt{p})^2>1$ which is not possible since $f:[0,1]\to[0,1]$. Thus $f(0)=0$, as claimed.

We claim that there exists $c\in[0,1]$ such that $f(x)=0$ for $0\le x\le c$ and $f(x)>0$ for $c<x<1$. Let $S=\big\{x\in[0,1]\big|f(x)=0\big\}$. Note that $S\ne\emptyset$ because $0\in S$ and $S$ is bounded above by 1. Let $c=\sup S$. Since $0\in S$ we have $c\ge 0$ and since 1 is an upper bound for $S$ we have $c\le 1$ and so $c\in[0,1]$. Since $c$ is an upper bound for $S$ it follows that $f(x)>0$ for all $x>c$ (when $x>c$ we must have $x\notin S$ and so $f(x)>0$). It remains to show that $f(x)=0$ for all $x\in[0,c]$. In the case that $c=0$ there is nothing to prove, so suppose that $c>0$. Suppose first that $x\in[0,c)$. By the Approximation Property we can choose $t\in S$ with $x<t\le c$. Since $t\in S$ we have $f(t)=0$. Since $f$ is increasing and $x<t$ we have $f(x)\le f(t)=0$ and so $f(x)=0$. This shows that $f(x)=0$ for all $x\in[0,c)$. Since $f$ is continuous at $c$, it follows that $f(c)=\lim\limits_{x\to c^-}f(x)=0$. Thus $f(x)=0$ for all $x\in[0,c]$, as required.

Let $c\in[0,1]$ be as above so that $f(x)=0$ for all $x\in[0,c]$ and $f(x)>0$ for all $x\in(c,1]$. When $c=1$ we have $f(x)=0$ for all $x\in[0,1]$. Suppose that $c<1$ and note that the interval $(c,1]$ is nonempty. As shown above, since $f(x)>0$ for all $x\in(c,1]$ we can choose $b\in\mathbb{R}$ so that $f(x)=(x-b)^2$ for all $x\in(c,1]$. Since $f(c)=0$ and $f$ is continuous at $c$ we have $0=f(c)=\lim\limits_{x\to c^+}f(x)=\lim\limits_{x\to c^+}(x-b)^2=(c-b)^2$ and hence we must have $b=c$. Thus $f(x)=0$ for $0\le x\le c$ and $f(x)=(x-c)^2$ for $c\le x\le 1$, as required.