PMATH 333, Exercises for Chapter 3

1: (a) Let $f(x)=\dfrac{8x}{2^{3x}}$ and let X be the partition of $[0,2]$ into 6 equal-sized subintervals. Find the Riemann sum for f on X which uses the right endpoints of the subintervals.

(b) Let $f(x)=\dfrac{1}{x}$ and let X be the partition of $\left[\frac{1}{5},\frac{13}{5}\right]$ into 6 equal-sized subintervals. Find the Riemann sum for f on X which uses the midpoints of the subintervals.

(c) Let $f(x)=4^{\cos x}$ and let $X=\left\{0.\frac{\pi}{3},\frac{\pi}{2},\frac{2\pi}{3},\pi,\frac{4\pi}{3},\frac{3\pi}{2},\frac{5\pi}{3},2\pi\right\}$. Find the average of the upper and lower Riemann sums for f on X.

2: (a) Suppose that f is increasing on $[a,b]$. Show that f is integrable on $[a,b]$.

(b) Suppose that $f(x)=0$ for all but finitely many points $x\in[a,b]$. Show that f is integrable on $[a,b]$.

(c) Define $f:[0,1]\to\mathbb{R}$ as follows. Let $f(0)=f(1)=0$. For $x\in(0,1)$ with $x\notin\mathbb{Q}$, let $f(x)=0$. For $x\in(0,1)$ with $x\in\mathbb{Q}$, write $x=\frac{a}{b}$ where $0<a,b\in\mathbb{Z}$ with $\gcd(a,b)=1$, and then let $f(x)=\frac{1}{b}$. Show that f is integrable in $[0,1]$.

3: (a) Let f be continuous with $f\geq 0$ on $[a,b]$. Show that if $\displaystyle\int_a^b f=0$ then $f=0$ on $[a,b]$.

(b) Find $g'(1)$ where $g(x)=\displaystyle\int_{3x-3}^{x^2+1}\sqrt{1+t^3}\,dt$.

(c) Find $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i}$.

4: (a) Let $0\leq a<b$. From the definition, show that $f(x)=x^2$ is integrable on $[a,b]$ with $\displaystyle\int_a^b f=\tfrac{1}{3}(b^3-a^3)$.

(b) Define $f:[1,2]\to\mathbb{R}$ by $f(x)=\begin{cases}x^2\ , \text{ if } x\notin\mathbb{Q}\\ 2x\ , \text{ if } x\in\mathbb{Q}\,.\end{cases}$ From the definition, show that $U(f)=3$ and $L(f)=\frac{7}{3}$.

5: (a) Find $\displaystyle\int_a^b x^3\,dx$ by evaluating the limit of a sequence of Riemann sums.

(b) Find $\displaystyle\int_0^8 \sqrt[3]{x}\,dx$ by evaluating the limit of a sequence of Riemann sums.

6: (a) Find $\displaystyle\int_1^2 \frac{1}{x}\,dx$ by evaluating the limit of a sequence of Riemann sums.

(b) Find $\displaystyle\int_1^2 \ln x\,dx$ by evaluating the limit of a sequence of Riemann sums.

7: (a) Find $\displaystyle\int_0^\pi \sin x\,dx$ by evaluating the limit of a sequence of Riemann sums.

(b) Find $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$ by evaluating the limit of a sequence of Riemann sums.

8: (a) Show that if f is integrable on $[a,b]$ then f^2 is integrable on $[a,b]$.

(b) Show that if f and g are both integrable on $[a,b]$, then fg is integrable on $[a,b]$.

(c) Show that if f is integrable and non-negative on $[a,b]$, then $\sqrt{f}$ is integrable on $[a,b]$.

9: Determine (with proof) which of the following statements are true.

(a) If $f:[a,b]\to[c,d]$ is integrable on $[a,b]$ and $g:[c,d]\to\mathbb{R}$ is integrable on $[c,d]$ then the composite $g\circ f$ must be integrable on $[a,b]$.

(b) If $f(x)=0$ for all but countably many $x\in[a,b]$ and $f(x)=1$ for countably many $x\in[a,b]$, then f cannot be integrable on $[a,b]$.

(c) If f is integrable on $[a,b]$ and the function $F(x)=\displaystyle\int_a^x f(t)\,dt$ is differentiable with $F'=f$ on $[a,b]$ then f is continuous on $[a,b]$.