PMATH 333 Real Analysis, Solutions to the Exercises for Chapter 3

1: (a) Let $f(x)=\dfrac{8x}{2^{3x}}$ and let X be the partition of $[0,2]$ into 6 equal-sized subintervals. Find the Riemann sum for f on X which uses the right endpoints of the subintervals.

Solution: The six intervals are of size $\Delta x=\frac{2-0}{6}=\frac{1}{3}$ and the right endpoints are the points $x_k=0+k\,\Delta x=\frac{k}{3}$, that is the points $\frac{1}{3},\frac{2}{3},1,\frac{4}{3},\frac{5}{3}$ and 2. We have

$$\begin{aligned}\sum_{k=1}^{n} f(x_i)\Delta x &= \left(f\left(\tfrac{1}{3}\right)+f\left(\tfrac{2}{3}\right)+f(1)+f\left(\tfrac{4}{3}\right)+f\left(\tfrac{5}{3}\right)+f(2)\right)\left(\tfrac{1}{3}\right)\\ &=\left(\tfrac{8\cdot1}{3\cdot2}+\tfrac{8\cdot2}{3\cdot4}+\tfrac{8\cdot3}{3\cdot8}+\tfrac{8\cdot4}{3\cdot16}+\tfrac{8\cdot5}{3\cdot32}+\tfrac{8\cdot6}{3\cdot64}\right)\left(\tfrac{1}{3}\right)\\ &=\left(\tfrac{4}{3}+\tfrac{4}{3}+\tfrac{3}{3}+\tfrac{2}{3}+\tfrac{5}{12}+\tfrac{3}{12}\right)\left(\tfrac{1}{3}\right)\\ &=\left(\tfrac{15}{3}\right)\left(\tfrac{1}{3}\right)\\ &=\tfrac{5}{3}\,.\end{aligned}$$

We remark that by using Integration by Parts, one can show that $\int_0^2 f(x)\,dx=\frac{21-2\ln 2}{24(\ln 2)^2}$.

(b) Let $f(x)=\dfrac{1}{x}$ and let X be the partition of $\left[\frac{1}{5},\frac{13}{5}\right]$ into 6 equal-sized subintervals. Find the Riemann sum for f on X which uses the midpoints of the subintervals.

Solution: The subintervals are of size $\Delta x=\frac{b-a}{n}=\frac{\frac{13}{5}-\frac{1}{5}}{6}=\frac{2}{5}$, and the endpoints are $x_k=a+\frac{b-a}{n}\,k=\frac{1}{5}+\frac{2}{5}\,k$ so that $x_0,x_1,x_2,\cdots x_6=\frac{1}{5},\frac{3}{5},\frac{5}{5},\cdots,\frac{13}{5}$, and the midpoints of the subintervals are $c_k=\frac{x_k+x_{k-1}}{2}$ so that $c_1,c_2,c_3,\cdots,c_6=\frac{2}{5},\frac{4}{5},\frac{6}{5},\cdots,\frac{12}{5}$. We have

$$\begin{aligned}\sum_{k=1}^{6} f(c_k)\Delta x &= \big(f(c_1)+f(c_2)+\cdots+f(c_6)\big)\left(\tfrac{2}{5}\right)\\ &=\tfrac{2}{5}\left(f\left(\tfrac{2}{5}\right)+f\left(\tfrac{4}{5}\right)+\cdots f\left(\tfrac{12}{5}\right)\right)\\ &=\tfrac{2}{5}\left(\tfrac{5}{2}+\tfrac{5}{4}+\tfrac{5}{6}+\cdots+\tfrac{5}{12}\right)\\ &=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}\\ &=\tfrac{60+30+20+15+12+10}{60}=\tfrac{147}{60}=\tfrac{49}{20}\,.\end{aligned}$$

We remark that $\displaystyle\int_{1/5}^{13/5} f(x)\,dx=\ln 13\,.$

(c) Let $f(x)=4^{\cos x}$ and let $X=\left\{0.\frac{\pi}{3},\frac{\pi}{2},\frac{2\pi}{3},\pi,\frac{4\pi}{3},\frac{3\pi}{2},\frac{5\pi}{3},2\pi\right\}$. Find the average of the upper and lower Riemann sums for f on X.

Solution: Note that $\cos x$ (and hence $f(x)$) is decreasing on $[0,\pi]$ and increasing on $[\pi.2\pi]$ and that $\cos x$ (hence $f(x)$) and the partition X are both symmetric about π, and so

$$\begin{aligned}U(f,X)&=2\Big(f(0)\cdot\tfrac{\pi}{3}+f\left(\tfrac{\pi}{3}\right)\cdot\tfrac{\pi}{6}+f\left(\tfrac{\pi}{2}\right)\cdot\tfrac{\pi}{6}+f\left(\tfrac{2\pi}{3}\right)\cdot\tfrac{\pi}{3}\Big)\\ &=2\Big(4\cdot\tfrac{\pi}{3}+2\cdot\tfrac{\pi}{6}+1\cdot\tfrac{\pi}{6}+\tfrac{1}{2}\cdot\tfrac{\pi}{3}\Big)=4\pi\end{aligned}$$

and

$$\begin{aligned}L(f,X)&=2\Big(f\left(\tfrac{\pi}{3}\right)\cdot\tfrac{\pi}{3}+f\left(\tfrac{\pi}{2}\right)\cdot\tfrac{\pi}{6}+f\left(\tfrac{2\pi}{3}\right)\cdot\tfrac{\pi}{6}+f(\pi)\cdot\tfrac{\pi}{3}\Big)\\ &=2\Big(2\cdot\tfrac{\pi}{3}+1\cdot\tfrac{\pi}{6}+\tfrac{1}{2}\cdot\tfrac{\pi}{6}+\tfrac{1}{4}\cdot\tfrac{\pi}{3}\Big)=2\pi\end{aligned}$$

and so the average of the upper and lower Riemann sums is 3π.

2: (a) Suppose that f is increasing on $[a,b]$. Show that f is integrable on $[a,b]$.

Solution: Suppose that f is increasing $\big($and hence bounded, below by $f(a)$ and above by $f(b)\big)$ on $[a,b]$. Notice that since f is increasing we have $M_k = f(x_k)$ and $m_k = f(x_{k-1})$, where $M_k = \sup\big\{f(t)\big|t\in[x_{k-1},x_k]\big\}$ and $m_k = \inf\big\{f(t)\big|t\in[x_{k-1},x_k]\big\}$, and so $\sum\limits_{k=1}^{n}(M_k - m_k) = \sum\limits_{k=1}^{n}\left(f(x_k)-f(x_{k-1})\right) = f(x_n)-f(x_0) = f(b)-f(a)$. Now let $\epsilon>0$. Choose a partition $X=\{x_0,x_1,\cdots,x_n\}$ of $[a,b]$ with $|X| < \frac{\epsilon}{f(b)-f(a)}$. Then

$$\begin{aligned}U(f,X)-L(f,X) &= \sum_{k=1}^{n} M_k\Delta_k x - \sum_{k=1}^{n} m_k\Delta_k x = \sum_{k=1}^{n}(M_k-m_k)\Delta_k x\\ &\leq \sum_{k=1}^{n}(M_k-m_k)|X| = \big(f(b)-f(a)\big)|X| < \epsilon.\end{aligned}$$

Thus f is integrable on $[a,b]$ (by Part 2 of Theorem 1.16).

(b) Suppose that $f(x)=0$ for all but finitely many points $x\in[a,b]$. Show that f is integrable on $[a,b]$.

Solution: Suppose that $f(x)=0$ except possibly at some of the points $p_0,p_1,p_2,\cdots,p_n$, where we have

$$a = p_0 < p_1 < \cdots < p_\ell = b\,.$$

Let $M = \max\big\{|f(p_k)|\,\big|\,0\leq k\leq \ell\big\}$. Let $\epsilon>0$ be arbitrary. Choose $\delta>0$ so that $\delta<\frac{\epsilon}{2\ell M}$ and so that $\delta<\frac{p_k-p_{k-1}}{2}$ (so that $p_{k-1}+\delta<p_k-\delta$) for all $k=1,2,\cdots,\ell$. Let X be the partition

$$X = \big\{p_0, p_0+\delta, p_1-\delta, p_1+\delta, p_2-\delta, p_2+\delta,\cdots,p_{\ell-1}-\delta,p_{\ell-1}+\delta,p_\ell-\delta,p_\ell\big\}\,.$$

For each $k=0,1,\cdots\ell$ let $M_k=\max\{f(p_k),0\}$ and let $m_k=\min\{f(p_k),0\}$. Note that $M_k-m_k=|f(p_k)|$, and we have

$$\begin{aligned}U(f,X) &= M_0\cdot\delta+0+M_1\cdot 2\delta+0+M_2\cdot2\delta+0+\cdots+M_{\ell-1}\cdot2\delta+0+M_\ell\cdot\delta\\ L(f,X) &= m_0\cdot\delta+0+m_1\cdot 2\delta+0+m_2\cdot2\delta+0+\cdots+m_{\ell-1}\cdot2\delta+0+m_\ell\cdot\delta\,.\end{aligned}$$

Thus

$$\begin{aligned}U(f,X)-L(f,X) &= (M_0-m_0)\cdot\delta+(M_1-m_1)\cdot2\delta+\cdots+(M_{\ell-1}-m_{\ell-1})\cdot2\delta+(M_\ell-m_\ell)\cdot\delta\\ &= \Big(|f(p_0)|+2|f(p_1)|+2|f(p_2)|+\cdots+2|f(p_{\ell-1})|+|f(p_\ell)|\Big)\cdot\delta\\ &\leq 2\ell M\,\delta<\epsilon\,.\end{aligned}$$

(c) Define $f:[0,1]\to\mathbb{R}$ as follows. Let $f(0)=f(1)=0$. For $x\in(0,1)$ with $x\notin\mathbb{Q}$, let $f(x)=0$. For $x\in(0,1)$ with $x\in\mathbb{Q}$, write $x=\frac{a}{b}$ where $0<a,b\in\mathbb{Z}$ with $\gcd(a,b)=1$, and then let $f(x)=\frac{1}{b}$. Show that f is integrable in $[0,1]$.

Solution: Let $\epsilon>0$ be arbitrary. Choose an integer $N>0$ so that $\frac{1}{N}<\frac{\epsilon}{2}$. Note that there are only finitely many points $x\in[0,1]$ such that $f(x)>\frac{1}{N}$ (indeed the only such points are the points $x=\frac{a}{b}$ with $0<a<b\in\mathbb{Z}$ with $b<N$). Say these points are $p_1,p_2,\cdots,p_{\ell-1}$ where

$$0=p_0<p_1<p_2<\cdots<p_{\ell-1}<p_\ell=1\,.$$

Choose $\delta>0$ so that $\delta<\frac{\epsilon}{2\ell}$ and so that $\delta<\frac{p_k-p_{k-1}}{2}$ for all $k=1,2,\cdots,\ell$. Let X be the partition

$$X=\big\{0,p_1-\delta,p_1+\delta,p_2-\delta,p_2+\delta,\cdots,p_{\ell-1}-\delta,p_{\ell-1}+\delta,1\big\}$$

Note that $L(f,X)=0$ and since $f(x)\leq\frac{1}{N}$ for all $x\neq p_k$, and $f(p_k)\leq\frac{1}{2}$ for all $k=1,2,\cdots,\ell-1$, we have

$$\begin{aligned}U(f,X) &\leq \tfrac{1}{N}(p_1-\delta)+f(p_1)\cdot2\delta+\tfrac{1}{N}(p_2-p_1-2\delta)+f(p_2)\cdot2\delta+\cdots+f(p_{\ell-1})\cdot2\delta+\tfrac{1}{N}(1-p_{\ell-1}-\delta)\\ &= \tfrac{1}{N}(1-2(\ell-1)\,\delta)+\big(f(p_1)+f(p_2)+\cdots+f(p_{\ell-1})\big)\cdot2\delta\\ &< \tfrac{1}{N}+\tfrac{\ell-1}{2}\cdot2\delta<\tfrac{1}{N}+\ell\,\delta<\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2}=\epsilon\,.\end{aligned}$$

3: (a) Let f be continuous with $f \geq 0$ on $[a,b]$. Show that if $\displaystyle\int_a^b f = 0$ then $f = 0$ on $[a,b]$.

Solution: Suppose that $f \neq 0$ on $[a,b]$. Choose $c \in [a,b]$ so that $f(c) \neq 0$. Note that $f(c) > 0$ since $f \geq 0$. Either $c \in [a,b)$ or $c \in (a,b]$. Let us suppose that $c \in [a,b)$ (the case $c \in (a,b]$ is similar). By the continuity of f we can choose $\delta > 0$ with $\delta < b - c$ so that for all $x \in [a,b]$ we have

$$|x-c| < \delta \Longrightarrow \big|f(x) - f(c)\big| < \tfrac{f(c)}{2} \Longrightarrow \tfrac{f(c)}{2} < f(x) < \tfrac{3f(c)}{2}\,.$$

Then by Additivity and Comparison we have

$$\begin{aligned}\int_a^b f &= \int_a^c f + \int_c^{c+\delta} f + \int_{c+\delta}^b f \\ &\geq \int_a^c 0 + \int_c^{c+\delta} \tfrac{f(c)}{2} + \int_{c-\delta}^b 0 \\ &= 0 + \tfrac{f(c)}{2}\,\delta + 0 > 0\,.\end{aligned}$$

(b) Find $g'(1)$ where $g(x) = \displaystyle\int_{3x-3}^{x^2+1} \sqrt{1+t^3}\,dt$.

Solution: Let $u(x) = x^2+1$ and let $v(x) = 3x-3$. Also, let $f(t) = \sqrt{1+t^3}$ and let $F(u) = \displaystyle\int_0^u \sqrt{1+t^3}\,dt$ so that $F'(u) = f(u)$, by the FTC. Then

$$g(x) = \int_{3x-3}^{x^2+1} \sqrt{1+t^3}\,dt = \int_0^{x^2+1} \sqrt{1+t^3}\,dt - \int_0^{3x-3} \sqrt{1+t^3}\,dt = F\big(u(x)\big) - F\big(v(x)\big)$$

and so $g'(x) = F'\big(u(x)\big)u'(x) - F'\big(v(x)\big)v'(x) = f\big(u(x)\big)(2x) - f\big(v(x)\big)(3) = 2x\,f(x^2+1) - 3\,f(3x-3)$. Put in $x = 1$ to get $g'(1) = 2f(2) - 3f(0) = 2\sqrt{1+8} - 3\sqrt{1+0} = 6 - 3 = 3$.

(c) Find $\displaystyle\lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n+i}$.

Solution: Let $f(x) = \frac{1}{1+x}$ and let X_n be the partition of $[0,1]$ into n equal-sized subintervals so $x_{n,k} = \frac{k}{n}$ and $\Delta_{n,k}x = \frac{1}{n}$. By recognizing a limit of Riemann sums as an integral, then applying the FTC, we have

$$\lim_{n\to\infty} \sum_{k=1}^n \frac{1}{n+k} = \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{1+\frac{k}{n}} \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{k=1}^n f(x_{n,k})\Delta_{n,k}x = \int_0^1 \frac{dx}{1+x} = \Big[\ln(1+x)\Big]_0^1 = \ln 2\,.$$

4: (a) Let $0 \le a < b$. From the definition, show that $f(x) = x^2$ is integrable on $[a,b]$ with $\int_a^b f = \frac{1}{3}(b^3 - a^3)$.

Solution: Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{2b(b-a)}$. Let X be any partition of $[a,b]$ with $|X| < \delta$. Let $t_k \in [x_{k-1}, x_k]$ be any sample points. Let $s_k = \sqrt{\frac{1}{3}\left(x_{k-1}{}^2 + x_{k-1}x_k + x_k{}^2\right)} \in [x_{k-1}, x_k]$. Note that $\sum_{k=1}^n f(s_k)\Delta_k x = \sum_{k=1}^n \frac{1}{3}\left(x_{k-1}{}^2 + x_{k-1}x_k + x_k{}^2\right)(x_k - x_{k-1}) = \sum_{k=1}^n \frac{1}{3}(x_k{}^3 - x_{k-1}{}^3) = \frac{1}{3}(b^3 - a^3)$, so

$$\left|\sum_{k=1}^n f(t_k)\Delta_k x - \tfrac{1}{3}(b^3 - a^3)\right| = \left|\sum_{k=1}^n f(t_k)\Delta_k x - \sum_{k=1}^n f(s_k)\Delta_k x\right| \le \sum_{k=1}^n \left|f(t_k) - f(s_k)\right|\Delta_k x$$
$$= \sum_{k=1}^n \left|t_k{}^2 - s_k{}^2\right|\Delta_k x = \sum_{k=1}^n |t_k + s_k||t_k - s_k|\Delta_k x < \sum_{k=1}^n 2b\,\delta\,\Delta_k x = \epsilon\,.$$

(b) Define $f : [1,2] \to \mathbb{R}$ by $f(x) = \begin{cases} x^2 \text{ , if } x \notin \mathbb{Q} \\ 2x \text{ , if } x \in \mathbb{Q}\,. \end{cases}$ From the definition, show that $U(f) = 3$ and $L(f) = \frac{7}{3}$.

Solution: First we shall show that $U(f) = 3$. To do this, we must show that for every partition X of $[1,2]$ we have $3 \le U(f,X)$, and also that for every $\epsilon > 0$ we can find a partition X of $[1,2]$ such that $U(f,X) - 3 < \epsilon$. Let $X = \{x_0, x_1, \cdots, x_n\}$ be any partition of $[1,2]$. Let $M_k = \sup\left\{f(t)\middle| t \in [x_{k-1}, x_k]\right\}$. Note that $M_k = 2x_k$ (since we can choose $t \in [x_{k-1}, x_k]$ arbitrarily close to x_k with $t \in \mathbb{Q}$ so that $f(t) = 2t$), so we have

$$U(f,X) = \sum_{k=1}^n M_k \Delta_k x = \sum_{k=1}^n 2x_k(x_k - x_{k-1}) \ge \sum_{k=1}^n (x_k + x_{k-1})(x_k - x_{k-1}) = \sum_{k=1}^n \left(x_k{}^2 - x_{k-1}{}^2\right)$$
$$= x_n{}^2 - x_0{}^2 = 2^2 - 1^2 = 3\,,$$

since the sum $\sum_{k=1}^n \left(x_k{}^2 - x_{k-1}{}^2\right)$ is a telescoping sum. Now let $\epsilon > 0$ be arbitrary. Choose a partition $X = \{x_0, x_1, \cdots, x_n\}$ with $|X| < \epsilon$. Let $M_k = \sup\left\{f(t)\middle| f(t) \in [x_{k-1}, x_k]\right\}$ Note, as above, that $M_k = 2x_k$ and that $\sum_{k=1}^n (x_k + x_{k-1})(x_k - x_{k-1}) = 3$, so we have

$$U(f,X) - 3 = \sum_{k=1}^n 2x_k \Delta_k x - \sum_{k=1}^n (x_k + x_{k-1})\Delta_k x = \sum_{k=1}^n (x_k - x_{k-1})\Delta_k x \le \sum_{k=1}^n |X|\Delta_k x < \sum_{k=1}^n \epsilon \Delta_k x = \epsilon\,.$$

To show that $L(f,X) = \frac{7}{3}$, we must show that for any partition X of $[1,2]$, we have $L(f,X) \le \frac{7}{3}$, and also that given any $\epsilon > 0$ there exists a partition X of $[1,2]$ such that $\frac{7}{3} - L(f,X) < \epsilon$. Let $X = \{x_0, x_1, \cdots, x_n\}$ be any partition of $[1,2]$. Let $s_k = \sqrt{\frac{1}{3}\left(x_{k-1}{}^2 + x_{k-1}x_k + x_k{}^2\right)}$. Note that, as shown in Part (a), we have $\sum_{k=1}^n s_k{}^2 \Delta_k x = \frac{1}{3}\left(2^3 - 1^3\right) = \frac{7}{3}$. Let $m_k = \inf\left\{f(t)\middle| t \in [x_{k-1}, x_k]\right\}$. Note that $m_k = x_{k-1}{}^2$ (since we can choose $t \in [x_{k-1}, x_k]$ arbitrarily close to x_{k-1} with $t \notin \mathbb{Q}$), and so

$$L(f,X) = \sum_{k=1}^n m_k \Delta_k x = \sum_{k=1}^n x_{k-1}{}^2 \Delta_k x \le \sum_{k=1}^n s_k{}^2 \Delta_k x = \tfrac{7}{3}\,.$$

Now let $\epsilon > 0$ be arbitrary. Choose a partition $X = \{x_0, x_1, \cdots, x_n\}$ of $[1,2]$ with $|X| < \frac{\epsilon}{3}$. As above, let $s_k = \sqrt{\frac{1}{3}\left(x_{k-1}{}^2 + x_{k-1}x_k + x_k{}^2\right)}$ so that $\sum_{k=1}^n s_k{}^2 \Delta_k x = \frac{7}{3}$, and let $m_k = \inf\left\{f(t)\middle| t \in [x_{k-1}, x_k]\right\} = x_{k-1}{}^2$. Then

$$\tfrac{7}{3} - L(f,X) = \sum_{k=1}^n s_k{}^2 \Delta_k x - \sum_{k=1}^n x_{k-1}{}^2 \Delta_k x = \sum_{k=1}^n \left(s_k{}^2 - x_{k-1}{}^2\right)\Delta_k x \le \sum_{k=1}^n \left(x_k{}^2 - x_{k-1}{}^2\right)\Delta_k x$$
$$\le \sum_{k=1}^n \left(x_k{}^2 - x_{k-1}{}^2\right)|X| < \frac{\epsilon}{3}\sum_{k=1}^n \left(x_k{}^2 - x_{k-1}{}^2\right) = \frac{\epsilon}{3}\left(2^2 - 1^2\right) = \epsilon\,.$$

5: (a) Find $\displaystyle\int_a^b x^3\,dx$ by evaluating the limit of a sequence of Riemann sums.

Solution: Let $f(x)=x^3$ and let $X_n=\big(x_{n,0},x_{n,1},\cdots,x_{n,n}\big)$ where $x_{n,k}=a+\frac{b-a}{n}\,k$ so $\Delta_{n,k}x=\frac{b-a}{n}$. Then

$$\begin{aligned}
\int_a^b x^3\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(x_{n,k})\Delta_{n,k}x\\
&= \lim_{n\to\infty}\sum_{k=1}^n \left(a+\tfrac{b-a}{n}\,k\right)^3\left(\tfrac{b-a}{n}\right)\\
&= \lim_{n\to\infty}\sum_{k=1}^n \Big(a^3+3a^2\left(\tfrac{b-a}{n}\right)k+3a\left(\tfrac{b-a}{n}\right)^2k^2+\left(\tfrac{b-a}{n}\right)^3k^3\Big)\left(\tfrac{b-a}{n}\right)\\
&= \lim_{n\to\infty}\sum_{k=1}^n \Big(a^3\left(\tfrac{b-a}{n}\right)\sum_{k=1}^n 1+3a^2\left(\tfrac{b-a}{n}\right)^2\sum_{k=1}^n k+3a\left(\tfrac{b-a}{n}\right)^3\sum_{k=1}^n k^2+\left(\tfrac{b-a}{n}\right)^4\sum_{k=1}^n k^4\Big)\\
&= \lim_{n\to\infty}\Big(a^3\left(\tfrac{b-a}{n}\right)n+3a^2\left(\tfrac{b-a}{n}\right)^2\tfrac{n(n+1)}{2}+3a\left(\tfrac{b-a}{n}\right)^3\tfrac{n(n+1)(2n+1)}{6}+\left(\tfrac{b-a}{n}\right)^4\tfrac{n^2(n+1)^2}{4}\Big)\\
&= a^3(b-a)+\tfrac{3}{2}\,a^2(b-a)^2+a(b-a)^3+\tfrac{1}{4}(b-a)^4\\
&= \tfrac{1}{4}(b-a)\big(4a^3+6a^2(b-a)+4a(b-a)^2+(b-a)^3\big)\\
&= \tfrac{1}{4}(b-a)(4a^3+6ab^2-6a^3+4ab^2-8a^2b+4a^3+b^3-3ab^2+3a^2b-a^3)\\
&= \tfrac{1}{4}(b-a)(a^3+a^2b+ab^2+b^3)\\
&= \tfrac{1}{4}(b^4-a^4)\,.
\end{aligned}$$

(b) Find $\displaystyle\int_0^8 \sqrt[3]{x}\,dx$ by evaluating the limit of a sequence of Riemann sums.

Solution: Let $f(x)=\sqrt[3]{x}$ and let $X_n=\big(x_{n,0},x_{n,1},\cdots,x_{n,n}\big)$ where $x_{n,k}=\left(\frac{2k}{n}\right)^3$. We have

$$\Delta_{n,k}x=x_{n,k}-x_{n,k-1}=\left(\tfrac{2k}{n}\right)^3-\left(\tfrac{2(k-1)}{n}\right)^3=\tfrac{8}{n^3}\big(k^3-(k-1)^3\big)=\tfrac{8}{n^3}(3k^2-3k+1)\,.$$

Note that $3k^2-3k+1$ is increasing for $k\geq 1$ $\big($since $g(x)=3x^2-3x+1$ is increasing for $x\geq-\frac{1}{2}\big)$ and so we have $|X_n|=\Delta_{n,n}x=\frac{8}{n^3}(3n^2-3n+1)\to 0$ as $n\to\infty$. Thus

$$\begin{aligned}
\int_0^\infty \sqrt[3]{x}\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(x_{n,k})\Delta_{n,k}x\\
&= \lim_{n\to\infty}\sum_{k=1}^n \left(\tfrac{2k}{n}\right)\left(\tfrac{8}{n^3}\right)(3k^2-3k+1)\\
&= \lim_{n\to\infty}\Big(\tfrac{48}{n^4}\sum_{k=1}^n k^3+\tfrac{48}{n^4}\sum_{k=1}^n k^2+\tfrac{16}{n^4}\sum_{k=1}^n k\Big)\\
&= \lim_{n\to\infty}\Big(\tfrac{48}{n^4}\,\tfrac{n^2(n+1)^2}{4}-\tfrac{48}{n^4}\,\tfrac{n(n+1)(2n+1)}{6}+\tfrac{16}{n^4}\,\tfrac{n(n+1)}{2}\Big)\\
&= \tfrac{48}{4}-0+0\\
&= 12\,.
\end{aligned}$$

6: (a) Find $\displaystyle\int_1^2 \frac{1}{x}\,dx$ by evaluating the limit of a sequence of Riemann sums.

Solution: Let $f(x)=\frac{1}{x}$ and let $X_n=\big(x_{n,0},x_{n,1},\cdots,x_{n,n}\big)$ with $x_{n,k}=2^{k/n}$. Note that

$$\Delta_{n,k}x=x_{n,k}-x_{n,k-1}=2^{k/n}-2^{(k-1)/n}=2^{k/n}\left(1-2^{-1/n}\right).$$

Since $2^{k/n}$ is increasing with k, we have $|X_n|=\Delta_{n,n}x=2\left(1-2^{-1/n}\right)\to 0$ as $n\to\infty$, and so

$$\begin{aligned}\int_1^2 \frac{1}{x}\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(x_{n,k})\Delta_{n,k}x=\lim_{n\to\infty}\sum_{k=1}^n 2^{-k/n}\,2^{k/n}\left(1-2^{-1/n}\right)\\ &=\lim_{n\to\infty}\left(1-2^{-1/n}\right)\sum_{k=1}^n=\lim_{n\to\infty}\left(1-2^{-1/n}\right)n=\lim_{n\to\infty}\frac{1-2^{-1/n}}{\frac{1}{n}}\\ &=\lim_{x\to 0}\frac{1-2^{-x}}{x}=\lim_{x\to 0}\frac{\ln 2\cdot 2^{-x}}{1}\quad\text{by l'Hospital's Rule}\\ &=\ln 2\,.\end{aligned}$$

(b) Find $\displaystyle\int_1^2 \ln x\,dx$ by evaluating the limit of a sequence of Riemann sums.

Solution: We shall need a formula for $S=\sum\limits_{k=1}^n k\,r^k$. We have

$$\begin{aligned}S&=1r+2r^2+3r^3+\cdots nr^n \text{ and}\\ rS&=1r^2+2r^3+\cdots+(n-1)r^n+nr^{n+1}\end{aligned}$$

so that

$$rS-S=nr^{n+1}-(r+r^2+r^3+\cdots+r^n)=nr^{n+1}-\frac{r^{n+1}-r}{r-1}=\frac{nr^{n+2}-nr^{n+1}-r^{n+1}-r}{r-1},$$

and hence

$$\sum_{k=1}^n k\,r^k=S=\frac{nr^{n+2}-(n+1)r^{n+1}-r}{(r-1)^2}.$$

Now let $f(x)=\ln x$ and let $X_n=\big(x_{n,0},x_{n,1},\cdots,x_{n,n}\big)$ with $x_{n,k}=e^{k\,\ln 2/n}=2^{k/n}$, as above. Then

$$\begin{aligned}\int_1^2 \ln x\,dx &= \lim_{n\to\infty}\sum_{k=1}^n f(x_{n,k})\Delta_{n,k}=\lim_{n\to\infty}\sum_{k=1}^n\left(\tfrac{k\,\ln 2}{n}\right)\left(2^{k/n}\right)\left(1-2^{-1/n}\right)\\ &=\lim_{n\to\infty}\left(\tfrac{\ln 2}{n}\right)\left(1-2^{-1/n}\right)\sum_{k=1}^n k\,\left(2^{1/n}\right)^k\\ &=\lim_{n\to\infty}\frac{\ln 2}{n}\cdot\frac{2^{1/n}-1}{2^{1/n}}\cdot\frac{2^{1/n}\left(n\,2^{(n+1)/n}-(n+1)2+1\right)}{(2^{1/n}-1)^2}\quad\text{, by the formula for }\sum_{k=1}^n k\,r^k\\ &=\lim_{n\to\infty}\frac{\ln 2\left(2^{(n+1)/n}-\frac{n+1}{n}\,2+\frac{1}{n}\right)}{2^{1/n}-1}=\lim_{n\to\infty}\frac{\ln 2\left(2\cdot 2^{1/n}-2-\frac{2}{n}+\frac{1}{n}\right)}{2^{1/n}-1}\\ &=\lim_{n\to\infty}\frac{\ln 2\left(2(2^{1/n}-1)-\frac{1}{n}\right)}{2^{1/n}-1}=\ln 2\left(2-\lim_{n\to\infty}\frac{\frac{1}{n}}{2^{1/n}-1}\right)\\ &=\ln 2\left(2-\lim_{x\to 0}\frac{x}{2^x-1}\right)=\ln 2\left(2-\lim_{x\to 0}\frac{1}{\ln 2\cdot 2^x}\right)\quad\text{, by l'Hospital's Rule}\\ &=\ln 2\left(2-\tfrac{1}{\ln 2}\right)=2\ln 2-1\,.\end{aligned}$$

7: (a) Find $\displaystyle\int_0^\pi \sin x\,dx$ by evaluating the limit of a sequence of Riemann sums.

Solution: Let $f(x)=\sin x$ and let X_n be the partition of $[0,\pi]$ into n equal-sized subintervals, so $x_{n,k}=\frac{\pi\,k}{n}$ and $\Delta_{n,k}x=\frac{\pi}{n}$. Then we have

$$\int_0^\pi \sin x\,dx=\lim_{n\to\infty}\sum_{k=1}^n f(x_{n,k})\Delta_{n,k}x=\lim_{n\to\infty}\sum_{k=1}^n \tfrac{\pi}{n}\,\sin\left(\tfrac{k\pi}{n}\right).$$

To find a formula for the sum $\displaystyle\sum_{k=1}^n \sin\left(\tfrac{k\pi}{n}\right)$, let $\alpha=e^{i\pi/n}$ so $\sin\frac{k\pi}{n}=\mathrm{Im}(\alpha^k)$. Note that $\alpha^n=-1$ and $\alpha\overline{\alpha}=1$, so we have

$$\begin{aligned}\sum_{k=1}^n \sin\tfrac{k\pi}{n}&=\mathrm{Im}\left(\sum_{k=1}^n\alpha^k\right)=\mathrm{Im}\left(\frac{\alpha(1-\alpha^n)}{1-\alpha}\right)=\mathrm{Im}\left(\frac{2\alpha}{1-\alpha}\right)=\mathrm{Im}\left(\frac{2\alpha(1-\overline{\alpha})}{(1-\alpha)(1-\overline{\alpha})}\right)\\&=\mathrm{Im}\left(\frac{2(\alpha-\alpha\overline{\alpha})}{1-2\mathrm{Re}(\alpha)+\alpha\overline{\alpha}}\right)=\mathrm{Im}\left(\frac{\alpha-1}{1-\mathrm{Re}(\alpha)}\right)=\frac{\mathrm{Im}(\alpha)}{1-\mathrm{Re}(\alpha)}=\frac{\sin\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}.\end{aligned}$$

Thus we have

$$\begin{aligned}\int_0^\pi \sin x\,d&=\lim_{n\to\infty}\sum_{k=1}^n\tfrac{\pi}{n}\,\sin\left(\tfrac{k\pi}{n}\right)=\lim_{n\to\infty}\frac{\frac{\pi}{n}\sin\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}=\lim_{x\to0}\frac{x\,\sin x}{1-\cos x}\\&=\lim_{x\to0}\frac{\sin x+x\cos x}{\sin x}\quad\text{, by l'Hospital's Rule}\\&=\lim_{x\to0}\frac{\cos x+\cos x-x\sin x}{\cos x}\quad\text{, by l'Hospital's Rule again}\\&=2\,.\end{aligned}$$

(b) Find $\displaystyle\int_0^1\sqrt{1-x^2}\,dx$ by evaluating the limit of a sequence of Riemann sums.

Solution: Let $f(x)=\sqrt{1-x^2}$. Let $X_n=\{x_{n,0},x_{n,1},\cdots,x_{n,n}\}$ where $x_{n,k}=\sin\left(\frac{k\pi}{2n}\right)$. We have

$$\begin{aligned}\Delta_{n,k}x&=\sin\left(\tfrac{k\pi}{2n}\right)-\sin\left(\tfrac{(k-1)\pi}{2n}\right)\\&=\sin\left(\tfrac{k\pi}{2n}\right)-\sin\left(\tfrac{k\pi}{2n}\right)\cos\left(\tfrac{\pi}{2n}\right)+\cos\left(\tfrac{k\pi}{2n}\right)\sin\left(\tfrac{k\pi}{2n}\right)\\&=\sin\left(\tfrac{k\pi}{2n}\right)\left(1-\cos\left(\tfrac{\pi}{2n}\right)\right)+\cos\left(\tfrac{k\pi}{2n}\right)\sin\left(\tfrac{\pi}{2n}\right).\end{aligned}$$

Note that $|X_n|\le\Delta_{n,k}x\le1-\cos\frac{\pi}{2n}+\sin\frac{\pi}{2n}\to0$ as $n\to\infty$. Using the formula $\displaystyle\sum_{k=1}^n\sin\tfrac{k\pi}{n}=\frac{\sin\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}$, which we derived in the solution to Part (a), and the formula $\displaystyle\sum_{k=1}^n\cos\tfrac{k\pi}{n}=-1$ (which could be derived in the same way as the previous formula, but can also be seen immediately using the symmetry $\cos\frac{k\pi}{n}=-\cos\frac{(n-k)\pi}{n}$), we have

$$\begin{aligned}\int_0^1\sqrt{1-x^2}\,dx&=\lim_{n\to\infty}\sum_{k=1}^n\sqrt{1-\sin^2\left(\tfrac{k\pi}{2n}\right)}\,\Delta_{n,k}x=\lim_{n\to\infty}\sum_{k=1}^n\left(\cos\tfrac{k\pi}{2n}\right)\left(\sin\tfrac{k\pi}{2n}\left(1-\cos\tfrac{\pi}{2n}\right)+\cos\tfrac{k\pi}{2n}\sin\tfrac{\pi}{2n}\right)\\&=\lim_{n\to\infty}\sum_{k=1}^n\left(\tfrac12\sin\tfrac{k\pi}{n}\left(1-\cos\tfrac{\pi}{2n}\right)+\tfrac12\left(1+\cos\tfrac{k\pi}{n}\right)\sin\tfrac{\pi}{2n}\right)\\&=\lim_{n\to\infty}\left(\tfrac12\left(1-\cos\tfrac{\pi}{2n}\right)\sum_{k=1}^n\sin\tfrac{k\pi}{n}+\tfrac12\sin\tfrac{\pi}{2n}\sum_{i=1}^n1+\tfrac12\sin\tfrac{\pi}{2n}\sum_{k=1}^n\cos\tfrac{k\pi}{n}\right)\\&=\lim_{n\to\infty}\left(\tfrac12\left(1-\cos\tfrac{\pi}{2n}\right)\frac{\sin\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}+\tfrac12 n\,\sin\tfrac{\pi}{2n}-\tfrac12\sin\tfrac{\pi}{2n}\right)\\&=0+\tfrac{\pi}{4}-0=\tfrac{\pi}{4}\text{ , where we used l'Hôpital's Rule.}\end{aligned}$$

8: (a) Show that if f is integrable on $[a,b]$ then f^2 is integrable on $[a,b]$.

Solution: Suppose that f is integrable on $[a,b]$. Then we know that $|f|$ is also integrable on $[a,b]$. Let M be an upper bound for $|f|$. Let $\epsilon>0$ be arbitrary. Choose a partition X of $[a,b]$ so that $U(|f|,X)-L(|f|,X)<\frac{\epsilon}{2M}$. Note that $M_k(f^2)=M_k(|f|)^2$ and $m_k(f^2)=m_k(|f|)^2$ so we have

$$\begin{aligned} M_k(f^2)-m_k(f^2) &= M_k(|f|)^2-m_k(|f|)^2 \\ &= \big(M_k(|f|)-m_k(|f|)\big)\big(M_k(|f|)+m_k(|f|)\big)\,. \\ &\leq \big(M_k(|f|)-m_k(|f|)\big)\cdot 2M \end{aligned}$$

Thus

$$\begin{aligned} U(f^2,X)-L(f^2,X) &= \sum_{k-1}^{n}\big(M_k(f^2)-m_k(f^2)\big)\Delta_k x \\ &\leq \sum_{k=1}^{n}\big(M_k(|f|)-m_k(|f|)\big)\cdot 2M\cdot\Delta_k x \\ &= 2M\big(U(|f|,X)-L(|f|,X)\big)<\epsilon\,. \end{aligned}$$

(b) Show that if f and g are both integrable on $[a,b]$, then fg is integrable on $[a,b]$.

Solution: Suppose that f and g are both integrable on $[a,b]$. Then, by linearity, $(f+g)$ is also integrable and so f^2, g^2 and $(f+g)^2$ are all integrable by part (a). Since $fg=\frac{1}{2}\big((f+g)^2-f^2-g^2\big)$, it is integrable too, by linearity.

(c) Show that if f is integrable and non-negative on $[a,b]$, then $\sqrt{f}$ is integrable on $[a,b]$.

Solution: Suppose that f is integrable and non-negative on $[a,b]$. When $X=\{x_0,x_1,\cdots,x_n\}$ is a partition of $[a,b]$, let us write $M_k(\sqrt{f})=\sup\big\{\sqrt{f(t)}\,\big|\,t\in[x_{k-1},x_k]\big\}$ and $M_k(f)=\sup\big\{f(t)\,\big|\,t\in[x_{k-1},x_k]\big\}$, and similarly for $m_k(\sqrt{f})$ and $m_k(f)$. Note that $M_k(f)=M_k(\sqrt{f})^2$ and $m_k(f)=m_k(\sqrt{f})^2$, and so we have

$$M_k(f)-m_k(f)=\big(M_k(\sqrt{f})-m_k(\sqrt{f})\big)\big(M_k(\sqrt{f})+m_k(\sqrt{f})\big)\,.$$

For any constant $c>0$, when $M_k(\sqrt{f})<c$ we have $M_k(\sqrt{f})-m_k(\sqrt{f})<c$, and when $M_k(\sqrt{f})>c$ we have $M_k(\sqrt{f})+m_k(\sqrt{f})>c$ so that $M_k(f)-m_k(f)\geq\big(M_k(\sqrt{f})-m_k(\sqrt{f})\big)\,c$, that is $M_k(\sqrt{f})-m_k(\sqrt{f})\leq\frac{1}{c}\big(M_k(f)-m_k(f)\big)$. Thus for any partition X and any constant $c>0$ we have

$$\sum_{k\text{ such that }M_k(\sqrt{f})<c}\big(M_k(\sqrt{f})-m_k(\sqrt{f})\big)\,\Delta_k x\;\leq\;\sum_{k=1}^{n}c\,\Delta_k x=c\,(b-a)\text{ , and}$$

$$\sum_{k\text{ such that }M_k(\sqrt{f})\geq c}\big(M_k(\sqrt{f})-m_k(\sqrt{f})\big)\,\Delta_k x\;\leq\;\sum_{k=1}^{n}\tfrac{1}{c}\big(M_k(f)-m_k(f)\big)\,\Delta_k x=\tfrac{1}{c}\big(U(f,X)-L(f,X)\big)\,.$$

Now, let $\epsilon>0$. Set $c=\dfrac{\epsilon}{2(b-a)}$ and choose a partition X of $[a,b]$ such that $U(f,X)-L(f,X)<\dfrac{\epsilon^2}{4(b-a)}$. Then

$$\begin{aligned} U(\sqrt{f},X)-L(\sqrt{f},X) &= \sum_{k=1}^{n}\big(M_k(\sqrt{f})-m_k(\sqrt{f})\big)\,\Delta_k x \\ &= \sum_{k\text{ with }M_k(\sqrt{f})<c}\big(M_k(\sqrt{f})-m_k(\sqrt{f})\big)\,\Delta_k x\;+\sum_{k\text{ with }M_k(\sqrt{f})\geq c}\big(M_k(\sqrt{f})-m_k(\sqrt{f})\big)\,\Delta_k x \\ &\leq\; c\,(b-a)+\tfrac{1}{c}\big(U(f,X)-L(f,X)\big) \\ &<\;\frac{\epsilon}{2(b-a)}\,(b-a)+\frac{2(b-a)}{\epsilon}\,\frac{\epsilon^2}{4(b-a)}=\epsilon\,. \end{aligned}$$

Thus $\sqrt{f}$ is integrable on $[a,b]$.

9: Determine (with proof) which of the following statements are true.

(a) If $f : [a,b] \to [c,d]$ is integrable on $[a,b]$ and $g : [c,d] \to \mathbb{R}$ is integrable on $[c,d]$ then the composite $g \circ f$ must be integrable on $[a,b]$.

Solution: This is false. Indeed let $f : [0,1] \to [0,1]$ be an integrable function with $f(x) > 0$ whenever $x \in \mathbb{Q}$ and $f(x) = 0$ whenever $x \notin \mathbb{Q}$, such as the function $f(x)$ from Problem 2(c), and let $g : [0,1] \to [0,1]$ be the map given by $g(0) = 0$ and $g(x) = 1$ for $x > 0$. We know that g is integrable on $[0,1]$ by Problem 2(b). But the composite function $g \circ f$ is not integrable on $[0,1]$, indeed we have $g(f(x)) = 0$ whenever $x \notin \mathbb{Q}$ and $g(f(x)) = 1$ whenever $x \in \mathbb{Q}$, and we have seen (in Example 1.4) that this function is not integrable.

(b) If $f(x) = 0$ for all but countably many $x \in [a,b]$ and $f(x) = 1$ for countably many $x \in [a,b]$, then f cannot be integrable on $[a,b]$.

Solution: This is false. Indeed, let

$$f(x) = \begin{cases} 1 \text{ if } x = 1 - \frac{1}{2^n} \text{ for some integer } n \geq 1, \\ 0 \text{ otherwise}. \end{cases}$$

We shall show that f is integrable on $[0,1]$. Let $\epsilon > 0$. We shall find a partition X of $[a,b]$ such that $U(f,X) - L(f,X) < \epsilon$. Choose n so that $\frac{n+1}{2^n} < \epsilon$ (we can do this since $\lim\limits_{n\to\infty} \frac{n+1}{2^n} = 0$, by l'Hôptal's Rule). For $k = 1, 2, \cdots, n$ let $x_k = 1 - \frac{1}{2^k} - \frac{1}{2^{n+1}}$ and $y_k = 1 - \frac{1}{2^k} + \frac{1}{2^{n+1}}$. Then $y_k - x_k = \frac{1}{2^n}$, and $x_k - y_{k-1} = \frac{1}{2^k} - \frac{1}{2^n}$, so for $k < n$ we have $x_k > y_{k-1}$ and we have $x_n = y_{n-1}$ and $y_n = 1 - \frac{1}{2^{n+1}}$. Let X be the partition $\{0, x_1, y_1, x_2, y_2, \cdots, x_{n-1}, y_{n-1} = x_n, y_n, 1\}$. On every subinterval, the minimum value of f is equal to 0, and so $L(f,X) = 0$. On each of the subintervals $[x_k, y_k]$, and also in the final subinterval $[y_n, 1]$, the maximum value of f is equal to 1, while in all the other subintervals, the maximum value of f is 0, and so

$$\begin{aligned} U(f,X) &= 0 + (y_1 - x_1) + 0 + (y_2 - x_2) + 0 + \cdots + 0 + (y_{n-1} - x_{n-1}) + (y_n - x_n) + (1 - y_n)) \\ &= n\,\tfrac{1}{2^n} + \tfrac{1}{2^{n+1}} < \tfrac{n+1}{2^n} < \epsilon\,. \end{aligned}$$

Thus $U(f,X) - L(f,X) < \epsilon$ as required.

(c) If f is integrable on $[a,b]$ and the function $F(x) = \displaystyle\int_a^x f(t)\,dt$ is differentiable with $F' = f$ on $[a,b]$ then f is continuous on $[a,b]$.

Solution: This is false. To find a counterexample, consider the function G given by $G(x) = x^2 \sin\frac{1}{x}$ when $x \neq 0$ and $G(0) = 0$. Note that G is differentiable. Let $f(x) = G'(x)$ for $x \in \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$, so we have $f(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Since f is continuous except at 0, f is integrable by part (a). We know, from the Fundamental Theorem, that the function $F(x) = \displaystyle\int_{-1/\pi}^x f(t)\,dt$ is continuous on $\left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$ and is differentiable with $F'(x) = f(x)$ for all $x \neq 0$. For $x < 0$ we have $F' = f = G'$ so $F = G + c_1$ for some constant c_1. Since $F\left(-\frac{1}{\pi}\right) = 0 = G\left(-\frac{1}{\pi}\right)$, we must have $c_1 = 0$, and so $F(x) = G(x)$ for all $x < 0$. Since F and G are both continuous at 0, we also have have $F(0) = G(0) = 0$. For $x > 0$ we again have $F' = f = G'$ so $F = G + c_2$ for some constant c_2. Since F and G are both continuous at 0 with $F(0) = G(0)$, we must have $c_2 = 0$ and so $F(x) = G(x)$ for all x. Thus F is differentiable with $F' = f$ for all x (including 0), but f is not continuous at 0.