PMATH 333 Real Analysis, Solutions to the Exercises for Chapter 4

1: (a) Define $f_n:[0,\infty)\to\mathbb{R}$ by $f_n(x)=nxe^{-nx}$. Find the pointwise limit $f(x)=\lim_{n\to\infty}f_n(x)$ and determine whether $f_n\to f$ uniformly on $[0,\infty)$.

Solution: Note that $f_n(0)=0$ hence $\lim_{n\to\infty}f_n(0)=0$. When $x>0$, we have $\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\dfrac{nx}{e^{nx}}=0$ by l'Hôpitals' Rule, indeed

$$\lim_{n\to\infty}\frac{nx}{e^{nx}}=\lim_{r\to\infty}\frac{rx}{e^{rx}}=\lim_{r\to\infty}\frac{\frac{d}{dr}(rx)}{\frac{d}{dr}(e^{rx})}=\lim_{r\to\infty}\frac{x}{xe^{rx}}=\lim_{r\to\infty}\frac{1}{e^{rx}}=0$$

since $e^{rx}\to\infty$ as $r\to\infty$. Thus the pointwise limit is $f(x)=\lim_{n\to\infty}nxe^{-nx}=0$ for all $x\in[0,\infty)$. In other words we have $f_n\to 0$ pointwise on $[0,\infty)$.

Note that $f_n\big(\frac{1}{n}\big)=\frac{1}{e}$ for all $n\in\mathbb{Z}^+$, and so the convergence is not uniform. To be very explicit, $f_n\to 0$ uniformly on $[0,\infty)$ means that $\forall\,\epsilon>0\ \exists\,m\in\mathbb{Z}^+\ \forall n\in\mathbb{Z}^+\ \forall x\in[0,\infty)\ \big(n\ge m\implies \big|f_n(x)-0\big|<\epsilon\big)$, so the convergence is not uniform when $\exists\,\epsilon>0\ \forall m\in\mathbb{Z}^+\ \exists n\in\mathbb{Z}^+\ \exists x\in[0,\infty)\ \big(n\ge m \text{ and } \big|f_n(x)-0\big|\ge\epsilon\big)$. To prove this, we choose $\epsilon=\frac{1}{e}$, we let $m\in\mathbb{Z}^+$, we choose $n=m$, and we choose $x=\frac{1}{n}$, and then we have $n\ge m$ and $\big|f_n(x)-0\big|=f_n\big(\frac{1}{n}\big)=\frac{1}{e}\ge\epsilon$.

(b) Define $f_n:[0,\infty)\to\mathbb{R}$ by $f_n(x)=\frac{x}{1+nx^2}$. Find the pointwise limit $f(x)=\lim_{n\to\infty}f_n(x)$ and determine whether $f_n\to f$ uniformly on $[0,\infty)$.

Solution: Note that $\lim_{n\to\infty}f_n(0)=\lim_{n\to\infty}0=0$, and when $x>0$ we have $\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\frac{x}{1+nx^2}=0$ since $1+nx^2\to\infty$ as $n\to\infty$. Thus the pointwise limit is $f(x)=\lim_{n\to\infty}\frac{x}{1+nx^2}=0$ for all $x\in[0,\infty)$. In other words, we have $f_n\to 0$ pointwise on $[0,\infty)$.

Let $n\in\mathbb{Z}^+$. Note that $f_n(x)=\frac{x}{1+nx^2}\ge 0$ for all $x\in[0,\infty)$ and we have $f_n{}'(x)=\frac{(1+nx^2)-x(2nx)}{(1+nx^2)^2}=\frac{1-nx^2}{(1+nx^2)^2}$ so that $f_n{}'(x)>0$ when $0\le x<\frac{1}{\sqrt{n}}$ and $f_n{}'(x)<0$ when $x>\frac{1}{\sqrt{n}}$. By the First Derivative Test, $f_n(x)$ attains its maximum value at $x=\frac{1}{\sqrt{n}}$ and the maximum value is $f_n\big(\frac{1}{\sqrt{n}}\big)=\frac{1}{2\sqrt{n}}$. Since $\big|f_n(x)\big|=f_n(x)\le\frac{1}{2\sqrt{n}}$ for all $x\in[0,\infty)$ and $\frac{1}{2\sqrt{n}}\to 0$ as $n\to\infty$, it follows that $f_n\to 0$ uniformly on $[0,\infty)$. To be very explicit, let $\epsilon>0$, choose $m\in\mathbb{Z}^+$ so that $\frac{1}{2\sqrt{m}}<\epsilon$, let $n\in\mathbb{Z}^+$ and let $x\in[0,\infty)$. Suppose that $n\ge m$. Then we have $\big|f_n(x)-0\big|=f_n(x)\le\frac{1}{2\sqrt{n}}\le\frac{1}{2\sqrt{m}}<\epsilon$.

(c) Define $f_n:[0,\infty]\to\mathbb{R}$ by $f_n(x)=\frac{x+n}{x+4n}$. Show that (f_n) converges uniformly on $[0,r]$ for every $r>0$ but that (f_n) does not converge uniformly on $[0,\infty)$.

Solution: The pointwise limit is $f(x)=\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\frac{x+n}{x+4n}=\frac{1}{4}$, so we have $f_n\to\frac{1}{4}$ pointwise on $[0,\infty)$. Note that $f_n{}'(x)=\frac{(x+4n)-(x+n)}{(x+4n)^2}=\frac{3n}{(x+4n)^2}>0$ for all $x\in[0,\infty)$ so that $f_n(x)$ is strictly increasing on $[0,\infty)$ with $f_n(0)=\frac{1}{4}$ and $\lim_{x\to\infty}f_n(x)=\lim_{x\to\infty}\frac{x+n}{x+4n}=1$. Because $\lim_{x\to\infty}f_n(x)=1$ it follows that (f_n) does not converge uniformly on $[0,\infty)$ to the constant function $\frac{1}{4}$. To be explicit, choose $\epsilon=\frac{1}{2}$, let $m\in\mathbb{Z}^+$, choose $n\ge m$, and choose $x\in[0,\infty)$ large enough so that $\big|f_n(x)-1\big|\le\frac{1}{4}$. Then we have $f_n(x)\ge 1-\frac{1}{4}=\frac{3}{4}$ so that $\big|f_n(x)-\frac{1}{4}\big|\ge\frac{3}{4}-\frac{1}{4}=\frac{1}{4}=\epsilon$.

On the other hand, we claim that for every $r>0$ we have $f_n\to\frac{1}{4}$ uniformly on $[0,r]$. Let $r>0$. Note that $f_n(0)=\frac{1}{4}$ and for $0<x\le r$ we have

$$\Big|f_n(x)-\frac{1}{4}\Big|=\Big|\frac{x+n}{x+4n}-\frac{1}{4}\Big|=\frac{3x}{4(x+4n)}=\frac{3}{4+\frac{16n}{x}}\le\frac{3}{4+\frac{16n}{r}}\longrightarrow 0 \text{ as } n\to\infty.$$

It follows that $f_n\to\frac{1}{4}$ uniformly on $[0,r]$, as claimed. Indeed, to be explicit, let $\epsilon>0$, choose $m\in\mathbb{Z}^+$ large enough that $\frac{3}{4+\frac{16m}{r}}<\epsilon$, let $x\in[0,r]$ and let $n\in\mathbb{Z}^+$ with $n\ge m$. Then $\big|f_n(x)-\frac{1}{4}\big|\le\frac{3}{4+\frac{16n}{r}}\le\frac{3}{4+\frac{16m}{r}}<\epsilon$.

2: (a) Find $\displaystyle\int_0^1 \lim_{n\to\infty} nx(1-x^2)^n\,dx$ and $\displaystyle\lim_{n\to\infty}\int_0^1 nx(1-x^2)^n\,dx$.

Solution: Let $x\in[0,1]$. If $x=0$ or $x=1$ then $nx(1-x^2)^n=0$ for all n and so $\lim\limits_{n\to\infty} nx(1-x^2)^n=0$. If $x\in(0,1)$ then $0<(1-x^2)<1$, so the series $\sum nx(1-x^2)^n$ converges by the Ratio Test and so $\lim\limits_{n\to\infty} nx(1-x^2)^n=0$ by the Divergence Test. Thus $\displaystyle\int_0^1 \lim_{n\to\infty} nx(1-x^2)^n\,dx=\int_0^1 0\,dx=0$. On the other hand, using the substitution $u=1-x^2$ so $du=-2x\,dx$ we have

$$\int_0^1 nx(1-x^2)^n\,dx=\int_1^0 -\tfrac{1}{2}n\,u^2\,du=\left[\frac{-n\,u^{n+1}}{2(n+1)}\right]_1^0=\frac{n}{2(n+1)}\,,$$

and so we have $\displaystyle\lim_{n\to\infty}\int_0^1 nx(1-x^2)^n\,dx=\tfrac{1}{2}$.

(b) Find $\displaystyle\int_1^4 \lim_{n\to\infty}\frac{\tan^{-1}(nx)}{x}\,dx$ and $\displaystyle\lim_{n\to\infty}\int_1^4 \frac{\tan^{-1}(nx)}{x}\,dx$.

Solution: Let $x\in[1,4]$. Then $\displaystyle\lim_{n\to\infty}\frac{\tan^{-1}(nx)}{x}=\frac{\pi}{2x}$ and so

$$\int_1^4 \lim_{n\to\infty}\frac{\tan^{-1}(nx)}{x}\,dx=\int_1^4\frac{\pi}{2x}\,dx=\Big[\tfrac{\pi}{2}\ln x\Big]_1^4=\pi\ln 2\,.$$

We claim that $\left\{\dfrac{\tan^{-1}(nx)}{x}\right\}\to\dfrac{\pi}{2x}$ uniformly on $[1,4]$. Indeed, given $\epsilon>0$ we can choose N so that $x\ge N\implies \big|\tan^{-1}x-\frac{\pi}{2}\big|<\epsilon$ for all $x\ge N$. Then for $n\ge N$ and $x\ge 1$ we have

$$\left|\frac{\tan^{-1}(nx)}{x}-\frac{\pi}{2x}\right|=\frac{\big|\tan^{-1}(nx)-\frac{\pi}{2}\big|}{x}<\frac{\epsilon}{x}\le\epsilon\,.$$

Since the convergence is uniform, $\displaystyle\lim_{n\to\infty}\int_1^4\frac{\tan^{-1}(nx)}{x}\,dx=\int_1^4\lim_{n\to\infty}\frac{\tan^{-1}(nx)}{x}\,dx=\pi\ln 2$.

(c) Show that $\displaystyle\sum_{n=0}^\infty\frac{\cos(2^nx)}{1+n^2}$ converges uniformly on $\mathbb{R}$ and find $\displaystyle\int_0^{\pi/4}\sum_{n=0}^\infty\frac{\cos(2^nx)}{1+n^2}\,dx$.

Solution: For all $x\in\mathbb{R}$ we have $\left|\dfrac{\cos(2^nx)}{1+n^2}\right|\le\dfrac{1}{1+n^2}<\dfrac{1}{n^2}$, and $\sum\frac{1}{n^2}$ converges, so $\displaystyle\sum_{n=0}^\infty\frac{\cos(2^nx)}{1+n^2}$ converges uniformly by the Weirstrass M-Test. Since the convergence is uniform,

$$\int_0^{\pi/4}\sum_{n=0}^\infty\frac{\cos(2^nx)}{1+n^2}\,dx=\sum_{n=0}^\infty\int_0^{\pi/4}\frac{\cos(2^nx)}{1+n^2}\,dx=\sum_{n=0}^\infty\left[\frac{1}{2^n}\,\frac{\sin(2^nx)}{1+n^2}\right]_0^{\pi/4}=\tfrac{\sqrt{2}}{2}+\tfrac14+0+0+\cdots=\tfrac{\sqrt{2}}{2}+\tfrac14.$$

(d) Show that $\displaystyle\sum_{n=1}^\infty\sin\left(\frac{x}{n^2}\right)$ converges uniformly on any closed interval $[a,b]$.

Solution: Note that $|\sin x|\le|x|$ for all $x\in\mathbb{R}$ and so $\left|\sin\left(\frac{x}{n^2}\right)\right|\le\frac{|x|}{n^2}$ for all x. Let $[a,b]$ be any closed interval and let $M=\max(|a|,|b|)$. Then for $x\in[a,b]$ we have $|x|\le M$ and so $\left|\sin\left(\frac{x}{n^2}\right)\right|\le\frac{|x|}{n^2}\le\frac{M}{n^2}$. Since $\sum\frac{M}{n^2}$ converges, $\sum\sin\left(\frac{x}{n^2}\right)$ converges uniformly on $[a,b]$ by the Weirstrass M-Test.

3: Determine which of the following statements are true for all sequences of functions (f_n) and (g_n) and all $E \subseteq \mathbb{R}$.

(a) If (f_n) and (g_n) converge uniformly on E then $(f_n g_n)$ converge uniformly on E.

Solution: This is FALSE. Let $E = \mathbb{R}$, let $f(x) = g(x) = x$ and let $f_n(x) = g_n(x) = x + \frac{1}{n}$. Then we have $f_n(x)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$ so $\lim\limits_{n\to\infty} f_n(x)^2 = x^2 = f(x)^2$ for all $x \in \mathbb{R}$, but the convergence is not uniform, since given any positive integer n, when $x \geq n$ we have $|f_n(x)^2 - f(x)^2| = \frac{2x}{n} + \frac{1}{n^2} > 2$.

(b) Show that if (f_n) and (g_n) converge uniformly on E and f and g are bounded on E then $(f_n g_n)$ converges uniformly on E.

Solution: This is TRUE. Suppose that (f_n) and (g_n) converge uniformly on E and f and g are bounded on E, say $|f(x)| \leq M$ and $|g(x)| \leq M$ for all $x \in E$. Choose N_1 so that $n \geq N_1 \Longrightarrow |f_n(x) - f(x)| < 1$. Note that for $n \geq N_1$ we have $|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq M + 1$. Now choose $N \geq N_1$ so that when $n \geq N$ we have $|f_n(x) - f(x)| < \frac{\epsilon}{2M}$ and $|g_n(x) - g(x)| < \frac{\epsilon}{2(M+1)}$ for all x. Then when $n \geq N$ we have

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| \\ &= |f_n(x)||g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)| \\ &\leq (M+1)\tfrac{\epsilon}{2(M+1)} + \tfrac{\epsilon}{2M}\, M = \epsilon\,. \end{aligned}$$

Thus $f_n g_n \to fg$ uniformly on E.

(c) If (f_n) converges uniformly on (a,b) and pointwise on $[a,b]$ then (f_n) converges uniformly on $[a,b]$.

Solution: This is TRUE. Indeed, suppose that (f_n) converges uniformly in (a,b) and that $\big(f_n(a)\big)$ and $\big(f_n(b)\big)$ both converge. Then given $\epsilon > 0$ we can choose N so that when $l, m \geq N$ we have $|f_l(x) - f_m(x)| < \epsilon$ for all $x \in (0,1)$, and $|f_l(a) - f_m(a)| < \epsilon$ and $|f_l(b) = f_m(b)| < \epsilon$, and so we have $|f_l(x) - f_m(x)| < \epsilon$ for all $x \in [a,b]$.

(d) If each f_n is continuous on $[a,b]$ and $\sum f_n$ converges uniformly on $[a,b]$ then $\sum M_n$ converges, where $M_n = \max\big\{|f_n(x)|\big|a \leq x \leq b\big\}$.

Solution: This is FALSE. For a counterexample, let

$$f_n(x) = \begin{cases} \frac{1}{n}\sin^2(2^n\pi\, x)\ , \text{ if } \frac{1}{2^n} \leq x \leq \frac{1}{2^{n-1}} \\ 0\ , \text{ otherwise.} \end{cases}$$

Then $M_n = \frac{1}{n}$ so $\sum M_n$ diverges, and yet we claim that $\sum f_n$ converges uniformly on $[0,1]$. Indeed if we write $S(x) = \sum\limits_{n=1}^{\infty} f_n(x)$ and $S_l(x) = \sum\limits_{n=l}^{\infty} f_n(x)$ then for all $x \in [0,1]$ we have

$$\big|S_l(x) - S(x)\big| = \sum_{n=l+1}^{\infty} f_n(x) \leq \max\{M_{l+1}, M_{l+2}, \cdots\} = \tfrac{1}{l+1}$$

since for each x, at most one of the terms $f_n(x)$ is non-zero.

4: (a) Find the Taylor series centered at 0, and its interval of convergence, for $f(x) = \dfrac{x}{x^2 - 6x + 8}$.

Solution: We have

$$f(x) = \frac{x}{x^2 - 6x + 8} = \frac{x}{(x-2)(x-4)} = \frac{-1}{x-2} + \frac{2}{x-4} = \frac{\frac{1}{2}}{1 - \frac{x}{2}} - \frac{\frac{1}{2}}{1 - \frac{x}{4}}\,.$$

Since $\dfrac{\frac{1}{2}}{1 - \frac{x}{2}} = \displaystyle\sum_{n=0}^{\infty} \tfrac{1}{2}\left(\tfrac{x}{2}\right)^n = \sum_{n=0}^{\infty} \tfrac{1}{2 \cdot 2^n} x^n$ when $\left|\frac{x}{2}\right| < 1$, that is $|x| < 2$, and $\dfrac{\frac{1}{2}}{1 - \frac{x}{4}} = \displaystyle\sum_{n=0}^{\infty} \tfrac{1}{2 \cdot 4^n} x^n$ when $|x| < 4$, we have

$$f(x) = \sum_{n=0}^{\infty} \tfrac{1}{2 \cdot 2^n} x^n - \sum_{n=0}^{\infty} \tfrac{1}{2 \cdot 4^n} x^n = \sum_{n=0}^{\infty} \tfrac{1}{2}\left(\tfrac{1}{2^n} - \tfrac{1}{4^n}\right) x^n$$

when $|x| < 2$.

(b) Find the Taylor series centered at $\frac{\pi}{4}$, and its interval of convergence, for $f(x) = \sin x \cos x$.

Solution: We provide two solutions. The first solution uses the known Taylor series for $\cos x$. We have

$$\begin{aligned} f(x) &= \sin x \cos x = \tfrac{1}{2} \sin 2x = \tfrac{1}{2} \cos\left(2x - \tfrac{\pi}{2}\right) = \tfrac{1}{2} \cos\left(2\left(x - \tfrac{\pi}{4}\right)\right) \\ &= \tfrac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(2\left(x - \tfrac{\pi}{4}\right)\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} \left(x - \tfrac{\pi}{4}\right)^{2n} \end{aligned}$$

for all $x \in \mathbb{R}$.

The second solution uses the formula for the coefficients of the Taylor series. We have $f(x) = \frac{1}{2} \sin 2x$, $f'(x) = \cos 2x$, $f''(x) = -2 \sin 2x$, $f'''(x) = -4 \cos 2x$, $f''''(x) = 8 \sin 2x$ and so on. Put in $x = \frac{\pi}{4}$ to get $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$, $f'\left(\frac{\pi}{4}\right) = 0$, $f''\left(\frac{\pi}{4}\right) = -2$, $f'''\left(\frac{\pi}{4}\right) = 0$, $f''''\left(\frac{\pi}{4}\right) = 8$ and so on. In general, the odd-order derivatives at 0 are all zero, that is $f^{(2n+1)}(0) = 0$, and the even-order derivatives are given by $f^{(2n)}(0) = (-1)^n 2^{2n-1}$. Thus the coefficients of the Taylor series are given by $c_{2n+1} = 0$ and $c_{2n} = \dfrac{(-1)^n 2^{2n-1}}{(2n)!}$, so the Taylor series is $T(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} \left(x - \tfrac{\pi}{4}\right)^{2n}$. To find the interval of convergence, let $a_n = \dfrac{(-1)^n 2^{2n-1}}{(2n)!} \left(x - \tfrac{\pi}{4}\right)^{2n}$. Then $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{4\left|x - \frac{\pi}{4}\right|^2}{(2n+2)(2n+1)} \to 0$ as $n \to \infty$, so $\sum a_n$ converges for all $x \in \mathbb{R}$.

(c) Let $0 < a < b$. Note that $\mathbb{Q} \cap [a, b]$ is countable, say $\mathbb{Q} \cap [a, b] = \{q_1, q_2, q_3, \cdots\}$. Find the interval of convergence of the power series $\sum\limits_{n=1}^{\infty} q_n x^n$.

Solution: Since $0 < a \le q_n \le b$, we have $0 < \sqrt[n]{a} \le \sqrt[n]{q_n} \le \sqrt[n]{b}$ for all n, and since $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1 = \lim\limits_{n\to\infty} \sqrt[n]{b}$ we have $\lim\limits_{n\to\infty} \sqrt[n]{q_n} = 1$ by the Squeeze Theorem. Thus the radius of convergence is $R = 1 \big/ \lim\limits_{n\to\infty} \sqrt[n]{q_n} = 1$. When $x = \pm 1$, $\lim\limits_{n\to\infty} q_n x^n$ does not exist and so $\sum q_n x^n$ diverges. Thus the interval of convergence is $I = (-1, 1)$.

5: (a) Find the 4$^{\text{th}}$ Taylor polynomial centered at 0 for $f(x) = \dfrac{\ln(1+x)}{e^{2x}}$.

Solution: We have

$$\begin{aligned} f(x) &= e^{-2x}\ln(1+x) \\ &= \Big(1 + (-2x) + \tfrac{1}{2!}(-2x)^2 + \tfrac{1}{3!}(-2x)^3 + \tfrac{1}{4!}(-2x)^4 + \cdots\Big)\Big(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots\Big) \\ &= \Big(1 - 2x + 2x^2 - \tfrac{4}{3}x^3 + \tfrac{2}{3}x^4 - \cdots\Big)\Big(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots\Big) \\ &= x - \big(\tfrac{1}{2} + 2\big)x^2 + \big(\tfrac{1}{3} + 1 + 2\big)x^3 - \big(\tfrac{1}{4} + \tfrac{2}{3} + 1 + \tfrac{4}{3}\big)x^4 + \cdots \\ &= x - \tfrac{5}{2}x^2 + \tfrac{10}{3}x^3 - \tfrac{13}{4}x^4 + \cdots \end{aligned}$$

so the Taylor polynomial of degree 4 is $T_4(x) = x - \frac{5}{2}x^2 + \frac{10}{3}x^3 - \frac{13}{4}x^4$.

(b) Find the 7$^{\text{th}}$ Taylor polynomial centered at 0 for $f(x) = \sec(\sqrt{2}\,x)$.

Solution: $f(x) = \dfrac{1}{\cos(\sqrt{2}\,x)} = \dfrac{1}{1 - \frac{1}{2}(2\,x^2) + \frac{1}{24}(4x^4) - \frac{1}{720}(8x^6) + \cdots} = \dfrac{1}{1 - x^2 + \frac{1}{6}x^4 - \frac{1}{90}x^6 + \cdots}$. We perform long division:

$$\begin{array}{r|l}
 & 1 + x^2 + \tfrac{5}{6}x^4 + \tfrac{61}{90}x^6 + \cdots \\ \hline
1 - x^2 + \tfrac{1}{6}x^4 - \tfrac{1}{90}x^6 + \cdots & 1 + 0x^2 + 0x^4 + 0x^6 + \cdots \\
 & 1 - x^2 + \tfrac{1}{6}x^4 - \tfrac{1}{90}x^6 + \cdots \\ \cline{2-2}
 & x^2 - \tfrac{1}{6}x^4 + \tfrac{1}{90}x^6 + \cdots \\
 & x^2 - x^4 + \tfrac{1}{6}x^6 + \cdots \\ \cline{2-2}
 & \tfrac{5}{6}x^4 - \tfrac{14}{90}x^6 + \cdots \\
 & \tfrac{5}{6}x^4 - \tfrac{5}{6}x^6 + \cdots \\ \cline{2-2}
 & \tfrac{61}{90}x^6 + \cdots
\end{array}$$

so $T_7(x) = 1 + x^2 + \frac{5}{6}x^4 + \frac{61}{90}x^6$.

(c) Let $f(x) = x^3 + x + 1$. Note that f is increasing with $f(0) = 1$, and let $g(x) = f^{-1}(x)$. Find the 6$^{\text{th}}$ Taylor polynomial centered at 1 for the inverse function $g(x)$.

Solution: Say $g(y) = a_0 + a_1(y-1) + a_2(y-1)^2 + a_3(y-1)^3 + \cdots$. Then

$$\begin{aligned} x = g\big(f(x)\big) &= g(x^3 + x + 1) = a_0 + a_1(x + x^3) + a_2(x + x^3)^2 + a_3(x + x^3)^3 + \cdots \\ &= a_0 + a_1(x + x^3) + a_2(x^2 + 2x^4 + x^6) + a_3\big(x^3 + 3x^5 + \cdots\big) \\ &\qquad + a_4\big(x^4 + 4x^6 + \cdots\big) + a_5\big(x^5 + \cdots\big) + a_6\big(x^6 + \cdots\big) + \cdots \\ &= a_0 + a_1x + a_2x^2 + (a_3 + a_1)x^3 + (a_4 + 2a_2)x^4 + (a_5 + 3a_3)x^5 + (a_6 + 4a_4 + a_2)x^6 + \cdots \end{aligned}$$

Comparing coefficients, we see that $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -a_1 = -1$, $a_4 = -2a_2 = 0$, $a_5 = -3a_3 = 3$ and $a_6 = -4a_4 - a_2 = 0$, and so the 6$^{\text{th}}$ Taylor polynomial is $T_6(x) = (x-1) - (x-1)^3 + 3(x-1)^5$.

6: (a) Let $f(x) = (8+x^3)^{2/3}$. Find $f^{(9)}(0)$, the 9^{th} derivative of f at 0.

Solution: $f(x) = (8+x^3)^{2/3} = 4\left(1+\frac{x^3}{8}\right)^{2/3} = 4\left(1 + \frac{2}{3}\frac{x^3}{8} + \frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{2!}\left(\frac{x^3}{8}\right)^2 + \frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}\left(\frac{x^3}{8}\right)^3 + \cdots\right)$, so $c_9 = \frac{4\cdot 2\cdot 1\cdot 4}{3^3 \cdot 3! \cdot 8^3} = \frac{1}{3^4\,2^5}$ and $f^{(9)}(0) = 9!\,c_9 = \frac{9!}{3^4\,2^5} = 140$.

(b) Evaluate the limit $\lim\limits_{x\to 0} \dfrac{x\,e^{x^2} - \sin x}{x - \tan^{-1} x}$.

Solution: $\lim\limits_{x\to 0} \dfrac{x\,e^{x^2} - \sin x}{x - \tan^{-1} x} = \lim\limits_{x\to 0} \dfrac{x\,(1+x^2+\frac{1}{2}x^4+\cdots) - (x - \frac{1}{6}x^3 + \cdots)}{x - (x - \frac{1}{3}x^3 + \cdots)} \lim\limits_{x\to 0} \dfrac{\frac{7}{6}x^3 + \cdots}{\frac{1}{3}x^3 + \cdots} = \frac{7}{2}$.

(c) Suppose that there exists a function $y = f(x)$, whose Taylor series centered at 0 has a positive radius of convergence, such that $\frac{1}{2}\,y'' + y' - 3y = x + 1$ with $y(0) = 1$ and $y'(0) = 2$. Find the Taylor polynomial of degree 5 centred at 0 for $f(x)$.

Solution: Let $y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots$. Then $y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots$ and $y'' = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots$. So we have

$$\begin{aligned} 0 &= \tfrac{1}{2}\,y'' + y' - 3y - x - 1 \\ &= (c_2 + 3c_3x + 6c_4x^2 + 10c_5x^3 + \cdots) + (c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots) \\ &\quad - (3c_0 + 3c_1x + 3c_2x^2 + 3c_3x^3 + \cdots) - x - 1 \\ &= (c_2 + c_1 - 3c_0 - 1) + (3c_3 + 2c_2 - 3c_1 - 1)x + (6c_4 + 3c_3 - 3c_2)x^2 + (10c_5 + 4c_4 - 3c_3)x^3 + \cdots \end{aligned}$$

Since $y(0) = 1$ and $y'(0) = 2$ we have $c_0 = 1$ and $c_2 = 2$. Put these values in the above equation to get

$$0 = (c_2 - 2) + (3c_3 + 2c_2 - 7)x + (6c_4 + 3c_3 - 3c_2)x^2 + (10c_5 + 4c_4 - 3c_3)x^3 + \cdots$$

For y to be a solution, all the coefficients must be zero, so we have

$$\begin{aligned} &(c_2 - 2) = 0 \implies c_2 = 2 \\ &(3c_3 + 2c_2 - 7) = 0 \implies 3c_3 = 7 - 2c_2 = 3 \implies c_3 = 1 \\ &(6c_4 + 3c_3 - 3c_2) = 0 \implies 6c_4 = 3c_2 - 3c_3 = 3 \implies c_4 = \tfrac{1}{2} \\ &(10c_5 + 4c_4 - 3c_3) = 0 \implies 10c_5 = 3c_3 - 4c_4 = 1 \implies c_5 = \tfrac{1}{10}\,. \end{aligned}$$

Thus the Taylor polynomial of degree 5 centered at 0 is

$$T_5(x) = 1 + 2x + 2x^2 + x^3 + \tfrac{1}{2}x^4 + \tfrac{1}{10}x^5\,.$$

7: Estimate each of the following numbers so that the error is at most $\frac{1}{1000}$.

(a) $\sqrt[5]{e}$

Solution: $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$, so we have

$$\sqrt[5]{e} = e^{\frac{1}{5}} = 1 + \tfrac{1}{5} + \tfrac{1}{5^2 2!} + \tfrac{1}{5^3 3!} + \tfrac{1}{5^4 4!} + \cdots \cong 1 + \tfrac{1}{5} + \tfrac{1}{5^2 2!} + \tfrac{1}{5^3 3!} = 1 + \tfrac{1}{5} + \tfrac{1}{50} + \tfrac{1}{750} = \tfrac{916}{750}$$

with error

$$\begin{aligned} E &= \tfrac{1}{5^4 4!} + \tfrac{1}{5^5 5!} + \tfrac{1}{5^6 6!} + \cdots = \tfrac{1}{5^4 4!}\left(1 + \tfrac{1}{5\cdot 5} + \tfrac{1}{5^2\cdot 5\cdot 6} + \tfrac{1}{5^3\cdot 5\cdot 6\cdot 7} + \cdots\right) \\ &\le \tfrac{1}{5^4 4!}\left(1 + \tfrac{1}{5^2} + \tfrac{1}{5^4} + \tfrac{1}{5^6} + \cdots\right) = \tfrac{1}{5^4 4!}\,\tfrac{1}{1-\frac{1}{25}} = \tfrac{1}{5^4 4!}\,\tfrac{25}{24} = \tfrac{1}{13200} \end{aligned}$$

where we used the C.T. and the formula for the sum of a geometric series.

(b) $\ln(4/5)$

Solution: We provide two solutions. For the first solution, we use $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$. We have

$$\ln\left(\tfrac{4}{5}\right) = \ln\left(1 - \tfrac{1}{5}\right) = -\tfrac{1}{5} - \tfrac{1}{2\cdot 5^2} - \tfrac{1}{3\cdot 5^3} - \tfrac{1}{4\cdot 5^4} - \cdots \cong -\tfrac{1}{5} - \tfrac{1}{2\cdot 5^2} - \tfrac{1}{3\cdot 5^3} = -\tfrac{1}{5} - \tfrac{1}{50} - \tfrac{1}{375} = -\tfrac{167}{750}$$

with error

$$E = \tfrac{1}{4\cdot 5^4} + \tfrac{1}{5\cdot 5^5} + \tfrac{1}{6\cdot 5^6} + \cdots < \tfrac{1}{4\cdot 5^4} + \tfrac{1}{4\cdot 5^5} + \tfrac{1}{4\cdot 5^6} + \cdots = \frac{\frac{1}{4\cdot 5^4}}{1 - \frac{1}{5}} = \tfrac{1}{4\cdot 5^4}\cdot\tfrac{5}{4} = \tfrac{1}{4^2\cdot 5^3} = \tfrac{1}{2000}\,,$$

where we used the C.T. and the formula for the sum of a geometric series.

For the second solution, we use $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$. We have

$$\ln\tfrac{4}{5} = -\ln\tfrac{5}{4} = -\ln\left(1 + \tfrac{1}{4}\right) = \left(-\tfrac{1}{4} + \tfrac{1}{2\cdot 4^2} - \tfrac{1}{3\cdot 4^3} + \tfrac{1}{4\cdot 4^4} - \cdots\right) \cong -\tfrac{1}{4} + \tfrac{1}{32} - \tfrac{1}{192} = -\tfrac{43}{192}$$

with error $E \le \frac{1}{4\cdot 4^4} < \frac{1}{1000}$ by the A.S.T.

(c) $\displaystyle\int_0^1 \sqrt{4 + x^3}\, dx$

Solution: Using the Binomial Series, we have

$$\begin{aligned} &\sqrt{4 + x^3}\, dx = 2\left(1 + \frac{x^3}{4}\right)^{1/2} \\ &= 2\left(1 + \tfrac{1}{2}\left(\frac{x^3}{4}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x^3}{4}\right)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x^3}{4}\right)^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}\left(\frac{x^3}{4}\right)^4 + \cdots\right) \\ &= 2 + \tfrac{1}{4}\,x^3 - \tfrac{1}{2\cdot 2!\cdot 4^2}\,x^6 + \tfrac{1\cdot 3}{2^2\cdot 3!\cdot 4^3}\,x^9 - \tfrac{1\cdot 3\cdot 5}{2^3\cdot 4!\cdot 4^4}\,x^{12} + \cdots \end{aligned}$$

and so

$$\begin{aligned} \int_0^1 \sqrt{4 + x^3}\, dx &= \Big[2\,x + \tfrac{1}{4\cdot 4}\,x^4 - \tfrac{1}{2\cdot 2!\cdot 4^2\cdot 7}\,x^7 + \tfrac{1\cdot 3}{2^2\cdot 3!\cdot 4^3\cdot 10}\,x^{10} - \tfrac{1\cdot 3\cdot 5}{2^3\cdot 4!\cdot 4^4\cdot 13}\,x^{13} + \cdots\Big]_0^1 \\ &= 2 + \tfrac{1}{4\cdot 4} - \tfrac{1}{2\cdot 2!\cdot 4^2\cdot 7} + \tfrac{1\cdot 3}{2^2\cdot 3!\cdot 4^3\cdot 10} - \tfrac{1\cdot 3\cdot 5}{2^3\cdot 4!\cdot 4^4\cdot 13} + \cdots \\ &\cong 2 + \tfrac{1}{4\cdot 4} - \tfrac{1}{2\cdot 2!\cdot 4^2\cdot 7} = 2 + \tfrac{1}{16} - \tfrac{1}{448} = \tfrac{923}{448} \end{aligned}$$

with absolute error $E \le \frac{1\cdot 3}{2^2\cdot 3!\cdot 4^3\cdot 10} = \frac{1}{5120}$ by the A.S.T.

To be rigorous, we should justify our application of the A.S.T. When $a_n = \dfrac{(-1)^{n+1}1\cdot 3\cdot 5\cdots(2n-3)}{2^{n-1}\cdot n!\cdot 4^n\cdot(3n+1)}$ we have $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{2n-1}{2\cdot(n+1)\cdot 4\cdot(3n+4)} < \dfrac{2n+2}{2\cdot(n+1)\cdot 4\cdot(3n+4)} = \dfrac{1}{4\cdot(3n+4)}$. Since $\left|\dfrac{a_{n+1}}{a_n}\right| < 1$ we know that $\{|a_n|\}$ is decreasing, and since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ we know that $\sum a_n$ converges by the R.T. so $\displaystyle\lim_{n\to\infty}|a_n| = 0$ by the D.T. Thus we can indeed apply the A.S.T.

8: Find the exact value of each of the following sums.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!}$

Solution: For all x we have $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $x\,e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$. Differentiate to get $(x+1)\,e^x = \displaystyle\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}$, so $(x^2+x)\,e^x = \displaystyle\sum_{n=0}^{\infty} \frac{(n+1)x^{n+1}}{n!}$. Differentiate again to get $(x^2+3x+1)\,e^x = \displaystyle\sum_{n=0}^{\infty} \frac{(n+1)^2x^n}{n!}$. Put in $x=1$ to get $5\,e = \displaystyle\sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(n+1)^2}{n!}$. Thus $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!} = 5\,e - 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2n+1)2^n}$

Solution: Let $S = \displaystyle\sum_{n=1}^{\infty} \frac{n}{(2n+1)2^n} = \frac{1}{3\cdot 2^1} + \frac{2}{5\cdot 2^2} + \frac{3}{7\cdot 2^3} + \cdots$. For $|x|<1$ we have $\dfrac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots$. Integrate both sides to get $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$. Divide both sides by x and the differentiate to get $\dfrac{\frac{x}{1-x^2} - \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}{x^2} = \frac{2x}{3} + \frac{4x^3}{5} + \frac{6x^5}{7} + \cdots$. Multiply by x to get $\frac{1}{1-x^2} - \frac{1}{2x}\ln\left(\frac{1+x}{1-x}\right) = \frac{2x^3}{3} + \frac{4x^4}{5} + \frac{6x^6}{7} + \cdots$. Put in $x = \frac{1}{\sqrt{2}}$ to get $2 - \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \frac{2}{3\cdot 2} + \frac{4}{5\cdot 2^2} + \frac{6}{7\cdot 2^3} + \cdots = 2S$. Thus

$$S = 1 - \tfrac{1}{2\sqrt{2}}\ln\left(\tfrac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 1 - \tfrac{1}{\sqrt{2}}\ln\left(\sqrt{2}+1\right)\,.$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3n-2}$

Solution: For $|x|<1$ we have $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{3n} = \frac{1}{1+x^3}$, so $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3n+1} = \int_0^x \frac{dt}{1+t^3}$. By Abel's Theorem we can put in $x=1$ to get $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} = \int_0^1 \frac{dt}{1+t^3}$. Thus

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3n-2} = \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n-2} = \tfrac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} = \tfrac{1}{2} + \int_0^1 \frac{dt}{t^3+1}\,.$$

To get $\dfrac{1}{t^3+1} = \dfrac{A}{t+1} + \dfrac{B(2t-1)+C}{t^2-t+1}$, we need $A(t^2-t+1) + B(2t^2+t-1) + C(t+1) = 1$. Equate coefficients to get the three equations $A+2B=0$, $-A+B+C=0$ and $A-B+C=1$. Solve these to get $A=\frac{1}{3}$, $B=-\frac{1}{6}$ and $C=\frac{1}{2}$. Thus we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3n-2} &= \tfrac{1}{2} + \int_0^1 \frac{dt}{t^3+1} = \tfrac{1}{2} + \int_0^1 \frac{\frac{1}{3}}{t+1} - \frac{\frac{1}{6}(2t-1)+\frac{1}{2}}{t^2-t+1}\,dt \\
&= \tfrac{1}{2} + \left[\tfrac{1}{3}\ln(t+1) - \tfrac{1}{6}\ln(t^2-t+1) + \tfrac{1}{\sqrt{3}}\tan^{-1}\tfrac{(t-\frac{1}{2})}{\frac{\sqrt{3}}{2}}\right]_0^1 \\
&= \tfrac{1}{2} + \tfrac{1}{3}\ln 2 + \tfrac{1}{\sqrt{3}}\tan^{-1}\tfrac{1}{\sqrt{3}} - \tfrac{1}{\sqrt{3}}\tan^{-1}\tfrac{1}{\sqrt{3}} = \tfrac{1}{2} + \tfrac{1}{3}\ln 2 + \tfrac{\pi}{3\sqrt{3}}\,.
\end{aligned}$$