PMATH 333, Exercises for Chapter 5

1: (a) Let $A = \text{Range}(f)$ where $f : \mathbb{R} \to \mathbb{R}^2$ is given by $f(t) = \big(\cos t, \sin 2t\big)$ and let $B = \text{Null}(g)$ where $g : \mathbb{R}^2 \to \mathbb{R}$ is given by $g(x,y) = y^2 + 4x^2(x^2-1)$. Prove (algebraically) that $A = B$.

(b) Let $f(x,y) = x^2 + 2y^2$ and $g(x,y) = 4x - y^2$. Find a parametric equation for the curve of intersection of the two surfaces $z = f(x,y)$ and $z = g(x,y)$.

2: (a) Let $A = \big\{(x,y) \in \mathbb{R}^2 \,\big|\, 0 < x,\ 0 < y \text{ and } xy < 1\big\}$. Show, from the definition of an open set, that A is open in $\mathbb{R}^2$.

(b) Let $B = \left\{ \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) \in \mathbb{R}^2 \,\middle|\, t \in \mathbb{R} \right\}$. Show that B is not closed in $\mathbb{R}^2$.

3: Let $A \subseteq \mathbb{R}^n$.

(a) Show that A' is closed in $\mathbb{R}^n$.

(b) Show that $\partial A = \overline{A} \setminus A^o$.

4: (a) Let $A, B \subseteq \mathbb{R}^n$ show that if A is connected and $A \subseteq B \subseteq \overline{A}$ then B is connected.

(b) Let S be a nonempty set and let $A_j \subseteq \mathbb{R}^n$ for each $j \in S$. Suppose that A_j is connected for all $j \in S$ and that $A_k \cap A_\ell \neq \emptyset$ for all $k, \ell \in S$. Show that $\bigcup_{j \in S} A_j$ is connected.

5: Let $A \subseteq P \subseteq \mathbb{R}^n$. Define the **interior of** A **in** P to be the union of all sets $E \subseteq P$ such that E is open in P and $E \subseteq A$. Define the **closure of** A **in** P to be the intersection of all sets $F \subseteq P$ such that F is closed in P and $A \subseteq F$. Denote the interior of A in $\mathbb{R}^n$ and the closure of A in $\mathbb{R}^n$ by A^o and $\overline{A}$ (as usual). Denote the interior of A in P and the closure of A in P by $\text{Int}_P(A)$ and $\text{Cl}_P(A)$.

(a) Show that $\text{Cl}_P(A) = \overline{A} \cap P$.

(b) Show that $\text{Int}_P(A) = (A \cup P^c)^o \cap P$, where $P^c = \mathbb{R}^n \setminus P$.

6: (a) Show, from the definition of compactness, that the set $A = \mathbb{Q} \cap [0,1]$ is not compact.

(b) Show, from the definition of compactness, that the set $B = \left\{ \frac{n|n|}{1+n^2} \,\middle|\, n \in \mathbb{Z} \right\} \cup \{1, -1\}$ is compact.

7: For each of the following functions $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ or show that the limit does not exist.

(a) $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

(b) $f(x,y) = \dfrac{x^2 y^3}{x^4 + y^6}$

(c) $f(x,y) = \dfrac{x^4 y^5}{x^8 + y^6}$

8: Let $f : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^m$.

(a) Show that f is continuous if and only if $f^{-1}(F)$ is closed in A for every closed set F in B.

(b) Let E and F be closed sets in A with $E \cup F = A$. Let g be the restriction of f to E, and let h be the restriction of f to F. Show that f is continuous if and only if both g and h are continuous.

(c) Show that f is continuous if and only if for every $E \subseteq A$ we have $f\big(\overline{E}\big) \subseteq \overline{f(E)}$.

9: (a) Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Show that if A is compact and f is continuous then f is uniformly continuous.

(b) Let $f : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^m$. Show that if A is compact and f is continuous and bijective then f^{-1} is continuous.

(c) Let $\emptyset \neq A, B \subseteq \mathbb{R}^n$. Define the **distance** between A and B to be

$$d(A,B) = \inf\big\{|x - y| \,\big|\, x \in A, y \in B\big\}.$$

Show that if A is compact and B is closed and $A \cap B = \emptyset$ then $d(A,B) > 0$.

10: Let $A \subseteq \mathbb{R}^n$.

(a) For $a, b \in A$, write $a \sim b$ when there exists a continuous path in A from a to b. Show that $\sim$ is an equivalence relation on A (this means that for all $a, b, c \in A$ we have $a \sim a$, and if $a \sim b$ then $b \sim a$, and if $a \sim b$ and $b \sim c$ then $a \sim c$).

(b) Suppose that A is open and connected. Show that A is path connected.