PMATH 333 Real Analysis, Solutions to the Exercises for Chapter 5

1: (a) Let $A = \text{Range}(f)$ where $f:\mathbb{R}\to\mathbb{R}^2$ is given by $f(t) = \big(\cos t, \sin 2t\big)$ and let $B = \text{Null}(g)$ where $g:\mathbb{R}^2\to\mathbb{R}$ is given by $g(x,y) = y^2 + 4x^2(x^2-1)$. Prove (algebraically) that $A = B$.

Solution: Note that $A = \text{Range}(f) = \big\{(\cos t, \sin 2t)\big| t\in\mathbb{R}\big\}$ and $B = \text{Null}(g) = \big\{(x,y)\big| y^2+4x^2(x^2-1)=0\big\}$. Let $(x,y)\in A$. Choose $t\in\mathbb{R}$ such that $x = \cos t$ and $y = \sin 2t$. Then $x^2 = \cos^2 t$ and

$$y^2 = 4\sin^2 t\cos^2 t = 4\cos^2 t(1-\cos^2 t) = 4x^2(1-x^2)$$

so we have $y^2 + 4x^2(x^2-1) = 0$ and so $(x,y)\in B$. Thus $A\subseteq B$.

Conversely, suppose that $(x,y)\in B$ so we have $y^2 = 4x^2(1-x^2)$. Then $y = \pm 2x\sqrt{1-x^2}$ with $-1\le x\le 1$. If $y = 2x\sqrt{1-x^2}$ then we can let $t = \cos^{-1}x \in [0,\pi]$, and then $\cos t = x$ and, since $\sin t\ge 0$,

$$\sin 2t = 2\sin t\cos t = 2\cos t\sqrt{\sin^2 t} = 2\cos t\sqrt{1-\cos^2 t} = 2x\sqrt{1-x^2} = y\,.$$

If $y = -2x\sqrt{1-x^2}$ then we can let $t = -\cos^{-1}x\in[-\pi,0]$, and then $\cos t = x$ and, since $\sin t\le 0$,

$$\sin 2t = 2\sin t\cos t = -2\cos t\sqrt{\sin^2 t} = -2\cos t\sqrt{1-\cos^2 t} = -2x\sqrt{1-x^2} = y\,.$$

In either case, we can choose $t\in\mathbb{R}$ such that $(x,y) = (\cos t, \sin 2t)$ and so $(x,y)\in A$. Thus $B\subseteq A$.

(b) Let $f(x,y) = x^2+2y^2$ and $g(x,y) = 4x - y^2$. Find a parametric equation for the curve of intersection of the two surfaces $z = f(x,y)$ and $z = g(x,y)$.

Solution: Set $f(x,y) = g(x,y)$ to get $x^2+2y^2 = 4x-y^2$, which we can write as $(x-2)^2+3y^2 = 4$. This is an ellipse, which we can parametrize as $(x,y) = \big(2+2\cos t, \frac{2}{\sqrt3}\sin t\big)$. We also need to have $z = 4x-y^2 = 8+8\cos t-\frac43\sin^2 t$, so a parametric equation for the curve of intersection is

$$(x,y,z) = \alpha(t) = \big(2+2\cos t, \tfrac{2}{\sqrt3}\sin t, 8+8\cos t - \tfrac43\sin^2 t\big).$$

To be rigorous, let us verify that $\text{Range}(\alpha) = \text{Graph}(f)\cap\text{Graph}(g)$. Let $(x,y,z)\in\text{Range}(\alpha)$. Choose $t\in\mathbb{R}$ such that $(x,y,z) = \alpha(t)$, so we have $x = 2+2\cos t$, $y = \frac{2}{\sqrt3}\sin t$ and $z = 8+8\cos t-\frac43\sin^2 t$. Then we have

$$f(x,y) = x^2+2y^2 = (2+2\cos t)^2 + 2\left(\tfrac{2}{\sqrt3}\sin t\right)^2 = 4+8\cos t+4\cos^2 t+\tfrac83\sin^2 t = 8+8\cos t-\tfrac43\sin^2 t = z$$

so that $(x,y,z)\in\text{Graph}(f)$, and we have

$$g(x,y) = 4x - y^2 = 4(2+2\cos t) - \left(\tfrac{2}{\sqrt3}\sin t\right)^2 = 8+8\cos t-\tfrac43\sin^2 t = z$$

so that $(x,y,z)\in\text{Graph}(g)$. Thus $\text{Range}(\alpha)\subseteq\text{Graph}(f)\cap\text{Graph}(g)$.

Let $(x,y,z)\in\text{Graph}(f)\cap\text{Graph}(g)$. Since $(x,y,z)\in\text{Graph}(f)$ we have $z = f(x,y) = x^2+2y^2$, and since $(x,y,z)\in\text{Graph}(g)$ we have $z = g(x,y) = 4x-y^2$. It follows that $x^2+2y^2 = 4x-y^2$, that is $(x-2)^2+3y^2 = 4$. Since $(x-2)^2 = 4-3y^2\le 4$ we have $\left|\frac{x-2}{2}\right|\le 1$. Since $3y^2 = 4-(x-2)^2\le 4$, we have $\left|\frac{\sqrt3}{2}y\right|\le 1$. Let $t\in[0,2\pi)$ be the (unique) angle with $\sin t = \frac{\sqrt3}{2}y$ and $\cos t = \frac{x-2}{2}$. Then we have $x = 2+2\cos t$, $y = \frac{2}{\sqrt3}\sin t$ and $z = g(x,y) = 4x-y^2 = 8+8\cos t-\frac43\sin^t$ and so $(x,y,z) = \alpha(t)\in\text{Range}(\alpha)$. Thus $\text{Graph}(f)\cap\text{Graph}(g)\subseteq\text{Range}(\alpha)$.

2: (a) Let $A=\{(x,y)\in\mathbb{R}^2 \,\big|\, 0<x,\ 0<y \text{ and } xy<1\}$. Show, from the definition of an open set, that A is open in $\mathbb{R}^2$.

Solution: Before beginning our proof, let us discuss our strategy. Suppose that $(a,b)\in A$, so we have $a>0$, $b>0$ and $ab<1$. We want to choose $r>0$ so that the disc $B_r=B\big((a,b),r\big)$ is contained in A. Note that the open square Q_r given by $|x-a|<r$ and $|y-b|<r$ contains the disc B_r, so it suffices to ensure that Q_r is contained in A. Note that if $r<a$ then $|x-a|<r \Longrightarrow |x-a|<a \Longrightarrow 0<x<2a \Longrightarrow x>0$. Similarly, if $r<b$ then $|y-b|<r \Longrightarrow y>0$. Note that if $r<a$ and $r<b$ then $r<a+b$ and so $(a+r)(b+r)=ab+r(a+b)+r^2<ab+r(a+b)+r(a+b)=ab+2r(a+b)$ and we can obtain $(a+r)(b+r)<1$ by choosing $r<\frac{1-ab}{2(a+b)}$.

Now we begin the proof. Let $(a,b)\in A$, so we have $a>0$, $b>0$ and $ab<1$. Choose $r=\min\big\{a,b,\frac{1-ab}{2(a+b)}\big\}$. Let $(x,y)\in B_r=B\big((a,b),r\big)$. Then $|x-a|=\sqrt{|x-a|^2}\le\sqrt{|x-a|^2+|y-b|^2}=\big|(x,y)-(a,b)\big|<r$ and similarly $|y-b|<r$. Since $|x-a|<r\le a$ we have $0\le a-r<x<a+r$ and since $|y-b|<r\le b$ we have $0\le b-r<y<b+r$. Since $0<x<a+r$ and $0<y<a+r$ and $r<a+b$ and $r<\frac{1-ab}{2(a+b)}$ we have $xy<(a+r)(b+r)=ab+r(a+b)+r^2<ab+2r(a+b)<ab+(1-ab)=1$. Since $x>0$ and $y>0$ and $xy<1$ we have $(x,y)\in A$. Thus $B_r\subseteq A$, as required, and so A is open.

(b) Let $B=\left\{\left(\frac{2t}{t^2+1},\frac{t^2-1}{t^2+1}\right)\in\mathbb{R}^2\,\middle|\, t\in\mathbb{R}\right\}$. Show that B is not closed in $\mathbb{R}^2$.

Solution: To solve this problem, you might find it helpful to draw a picture of the set B by choosing various values of t and plotting points. You should find that B looks like the unit circle centred at $(0,0)$ with the point $(0,1)$ removed. If you wish, you can show, algebraically, that this is indeed the case.

Let $a=(0,1)$. Let $x(t)=\frac{2t}{t^2+1}$ and $y(t)=\frac{t^2-1}{t^2+1}$ and $f(t)=\big(x(t),y(t)\big)$ so that $B=\big\{f(t)\big|t\in\mathbb{R}\big\}$. We claim that $a\in B'$ (that is a is a limit point of B) but $a\notin B$. It is clear that $a\notin B$ because to get $f(t)=a$ we need $x(t)=0$ and $y(t)=1$, but to get $x(t)=\frac{2t}{t^2+1}=0$ we must choose $t=0$, and then $y(t)=\frac{t^2-1}{t^2+1}=-1\ne 1$. To show that $a\in B'$, we shall show that for all $r>0$ we have $B(a,r)\cap B\ne\emptyset$. Let $r>0$. Since $\lim\limits_{t\to\infty}x(t)=0$ and $\lim\limits_{t\to\infty}y(t)=1$ we can choose $t\in\mathbb{R}$ so that $\big|x(t)-0\big|<\frac{r}{2}$ and $\big|y(t)-1\big|<\frac{r}{2}$. Then we have

$$\big|f(t)-a\big|=\big|(x(t),y(t))-(0,1)\big|=\big|\big(x(t)\,,\,y(t)-1\big)\big|\le|x(t)|+|y(t)-1|<\tfrac{r}{2}+\tfrac{r}{2}=r$$

and so $f(t)\in B(a,r)\cap B$. This shows that for all $r>0$ we have $B(a,r)\cap B\ne\emptyset$, and so $a\in B'$. Since $a\in B'$ and $a\notin B$ we do not have $B'\subseteq B$ and so B is not closed (by Part (2) of Theorem 5.19).

3: Let $A\subseteq\mathbb{R}^n$.

(a) Show that A' is closed in $\mathbb{R}^n$.

Solution: By Part (2) of Theorem 5.19, we know that A' is closed if and only if $(A')'\subseteq A'$. Let $a\in(A')'$, that is let a be a limit point of A'. Let $r>0$. Since a is a limit point of A', we know that $B^*(a,r)\cap A'\ne\emptyset$. Choose $b\in B^*(a,r)\cap A'$. Note that $0<|a-b|<r$. Let $s=\min\big(|a-b|\,,r-|a-b|\big)>0$. Since $b\in A'$ we know that $B^*(b,s)\cap A\ne\emptyset$. Choose $c\in B^*(b,s)\cap A$. We claim that $c\in B^*(a,r)\cap A$. By the Triangle Inequality we have $|a-c|\le|a-b|+|b-c|<|a-b|+s\le|a-b|+r-|a-b|=r$, and by the Triangle Inequality again, we have $|a-b|\le|a-c|+|c-b|$ and so $|a-c|\ge|a-b|-|b-c|>|a-b|-s\ge|a-b|-|a-b|=0$. Thus $0<|a-c|<r$ and so $c\in B^*(a,r)\cap A$, as claimed. Since $c\in B^*(a,r)\cap A$, we see that $B^*(a,r)\cap A\ne\emptyset$. We have shown that for every $r>0$ we have $B^*(a,r)\cap A\ne\emptyset$, and so $a\in A'$. This proves that $(A')'\subseteq A'$, and so A' is closed.

(b) Show that $\partial A=\overline{A}\setminus A^o$.

Solution: Let $a\in\partial A$. We claim first that $a\in\overline{A}$. Since $\overline{A}=A\cup A'$ it suffices to show that either $a\in A$ or $a\in A'$. Suppose that $a\notin A$. Let $r>0$ be arbitrary. Since $a\in\partial A$ we have $B(a,r)\cap A\ne\emptyset$. Since $a\notin A$ we have $B^*(a,r)\cap A=B(a,r)\cap A$ and so $B^*(a,r)\cap A)\ne\emptyset$. Since $r>0$ was arbitrary, we have $a\in A'$, as required.

Next we claim that $a\notin A^0$. Suppose, for a contradiction, that $a\in A^0$. By Part (b), a is an interior point of A so we can choose $r>0$ so that $B(a,r)\subseteq A$. Since $B(a,r)\subseteq A$ we have $B(a,r)\cap A^c=\emptyset$. But since $a\in\partial A$ we have $B(a,r)\cap A^c\ne\emptyset$, so we have obtained the desired contradiction. Thus $a\notin A^0$, as claimed. This completes the proof that $\partial A\subseteq\overline{A}\setminus A^0$.

Now let $a\in\overline{A}\setminus A^0$, that is let $a\in\overline{A}$ with $a\notin A^0$. Let $r>0$ be arbitrary. Case 1: suppose that $a\in A$. Let $r>0$ be arbitrary. Since $a\in A$ and $a\in B(a,r)$ we have $B(a,r)\cap A\ne\emptyset$. Since $a\notin A^0$ we have $B(a,r)\not\subseteq A$ and so $B(a,r)\cap A^c\ne\emptyset$. Thus $a\in\partial A$. Case 2: suppose that $a\notin A$. Let $r>0$ be arbitrary. Since $a\notin A$ and $a\in B(a,r)$ we have $B(a,r)\cap A^c\ne\emptyset$. Since $a\in\overline{A}=A\cup A'$ and $a\notin A$ we have $a\in A'$ and so $B^*(a,r)\cap A\ne\emptyset$ hence $B(a,r)\cap A\ne\emptyset$. Thus $a\in\partial A$. In either case we find that $a\in\partial A$. This completes the proof that $\overline{A}\setminus A^0\subseteq\partial A$.

4: (a) Let $A, B \subseteq \mathbb{R}^n$ show that if A is connected and $A \subseteq B \subseteq \overline{A}$ then B is connected.

Solution: Suppose that A is connected and that $A \subseteq B \subseteq \overline{A}$. Suppose, for a contradiction, that B is disconnected. Choose open sets $U, V \subseteq \mathbb{R}^n$ which separate B, so we have $U \cap B \neq \emptyset$, $V \cap B \neq \emptyset$, $U \cap V = \emptyset$ and $B \subseteq U \cup V$. We claim that U and V also separate A (contradicting the fact that A is connected). Since $A \subseteq B \subseteq U \cup V$, it suffices to prove that $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$. We claim that $U \cap A \neq \emptyset$. Since $U \cap B \neq \emptyset$ we can choose $b \in U \cap B$. Then we have $b \in B \subseteq \overline{A} = A \cup A'$, and so either $b \in A$ or $b \in A'$. If $b \in A$ then we have $b \in U \cap A$ so that $U \cap A \neq \emptyset$. Suppose that $b \in A'$. Since $b \in U$ and U is open, we can choose $r > 0$ such that $B(b, r) \subseteq U$. Since $b \in A'$ we have $B(b, r) \cap A \neq \emptyset$ so we can choose $c \in B(b, r) \cap A$. Then we have $c \in B(b, r) \subseteq U$ and $c \in A$, hence $c \in U \cap A$, and so $U \cap A \neq \emptyset$. This proves that $U \cap A \neq \emptyset$, as claimed. The proof that $V \cap A \neq \emptyset$ is similar, and so U and V separate A giving the desired contradiction.

(b) Let S be a nonempty set and let $A_j \subseteq \mathbb{R}^n$ for each $j \in S$. Suppose that A_j is connected for all $j \in S$ and that $A_k \cap A_\ell \neq \emptyset$ for all $k, \ell \in S$. Show that $\bigcup_{j \in S} A_j$ is connected.

Solution: Let $B = \bigcup_{j \in S} A_j$. Suppose, for a contradiction, that B is disconnected. Choose open sets $U, V \subseteq \mathbb{R}^n$ which separate B, that is $B \cap U \neq \emptyset$, $B \cap V \neq \emptyset$, $U \cap V = \emptyset$ and $B \subseteq U \cup V$. Choose $a \in B \cap U$ and $b \in B \cap V$. Since $a \in B = \bigcup_{j \in S} A_j$, we can choose $k \in S$ such that $a \in A_k$. Similarly we can choose $\ell \in S$ such that $b \in A_\ell$. Then we have $a \in A_k \cap U$ and $b \in A_\ell \cap V$. Since A_k is connected, and $a \in A_k \cap U$ so that $A_k \cap U \neq \emptyset$, and $A_k \subseteq \bigcup_{j \in S} A_j = B \subseteq U \cup V$, it follows that we must have $A_k \subseteq U$ because otherwise we would have $A_k \cap V \neq \emptyset$ and so U and V would separate A_k. Similarly, we must have $A_\ell \subseteq V$. Since $A_k \subseteq U$ and $A_\ell \subseteq V$ we have $A_k \cap A_\ell \subseteq U \cap V = \emptyset$. This contradicts our assumption that $A_k \cap A_\ell \neq \emptyset$, and so B is connected, as required.

5: Let $A \subseteq P \subseteq \mathbb{R}^n$. Define the **interior of** A **in** P to be the union of all sets $E \subseteq P$ such that E is open in P and $E \subseteq A$. Define the **closure of** A **in** P to be the intersection of all sets $F \subseteq P$ such that F is closed in P and $A \subseteq F$. Denote the interior of A in $\mathbb{R}^n$ and the closure of A in $\mathbb{R}^n$ by A^o and $\overline{A}$ (as usual). Denote the interior of A in P and the closure of A in P by $\text{Int}_P(A)$ and $\text{Cl}_P(A)$.

(a) Show that $\text{Cl}_P(A) = \overline{A} \cap P$.

Solution: Since $\overline{A}$ is closed in $\mathbb{R}^n$ it follows that $\overline{A} \cap P$ is closed in P. Since $A \subseteq \overline{A}$ and $A \subseteq P$ we have $A \subseteq \overline{A} \cap P$. Since $\overline{A} \cap P$ is closed in P and $A \subseteq \overline{A} \cap P$, it follows from the definition of $\text{Cl}_P(A)$ that $\text{Cl}_P(A) \subseteq \overline{A} \cap P$.

Let F be any closed set in P with $A \subseteq F$. Choose a closed set K in $\mathbb{R}^n$ such that $F = K \cap P$. Since K is closed in $\mathbb{R}^n$ and $A \subseteq K$ we have $\overline{A} \subseteq K$. Thus $\overline{A} \cap P \subseteq K \cap P = F$. Since $\overline{A} \cap P \subseteq F$ for every closed set F in P which contains A, it follows, from the definition of $\text{Cl}_P(A)$, that $\overline{A} \cap P \subseteq \text{Cl}_P(A)$.

(b) Show that $\text{Int}_P(A) = (A \cup P^c)^o \cap P$, where $P^c = \mathbb{R}^n \setminus P$.

Solution: Let $F = (A \cup P^c)^o \cap P$. Since $(A \cup P^c)^o$ is open in $\mathbb{R}^n$ it follows that $F = (A \cup P^c)^o \cap P$ is open in P. Also note that we have $F = (A \cup P^c)^o \cap P \subseteq (A \cup P^c) \cap P = (A \cap P) \cup (P^c \cap P) = (A \cap P) \cup \emptyset = A \cap P = A$, since $A \subseteq P$. Since F is open in P and $F \subseteq A$ it follows, from the definition of $\text{Int}_P(A)$, that $F \subseteq \text{Int}_P(A)$.

Let E be any open set in P with $E \subseteq A$. Choose an open set U in $\mathbb{R}^n$ such that $U \cap P = E$. Then we have $U = U \cap \mathbb{R}^n = U \cap (P \cup P^c) = (U \cap P) \cup (U \cap P^c) = E \cup (U \cap P^c) \subseteq A \cup P^c$, since $E \subseteq A$ and $U \cap P^c \subseteq P^c$. Since U is open in $\mathbb{R}^n$ and $U \subseteq A \cup P^c$ it follows that $U \subseteq (A \cup P^c)^o$. Since $E = U \cap P \subseteq U \subseteq (A \cup P^c)^o$ and $E \subseteq A \subseteq P$ we have $E \subseteq (A \cup P^c)^o \cap P = F$. Since $E \subseteq F$ for every open set E in P with $E \subseteq A$ it follows, from the definition of $\text{Int}_P(A)$, that $\text{Int}_P(A) \subseteq F$.

6: (a) Show, from the definition of compactness, that the set $A=\mathbb{Q}\cap[0,1]$ is not compact.

Solution: Let $a\in[0,1]$ with $a\notin\mathbb{Q}$ and note that a is a limit point of A because $\mathbb{Q}$ is dense in $\mathbb{R}$. For each $n\in\mathbb{Z}^+$ let $U_n=\overline{B}\big(a,\frac{1}{n}\big)^c=\big(-\infty,a-\frac{1}{n}\big)\cup\big(a+\frac{1}{n},\infty\big)$, and let $S=\big\{U_n\big|n\in\mathbb{Z}^+\big\}$. Note that each U_n is open and we have $\bigcup\limits_{n=1}^{\infty}U_n=\mathbb{R}\setminus\{a\}$, so S is an open cover of A. Let T be any nonempty finite subset of A, say $T=\big\{U_{n_1},U_{n_2},\cdots,U_{n_\ell}\big\}$ with $n_1<n_2<\cdots<n_\ell$. Note that $U_1\subseteq U_2\subseteq U_3\subseteq\cdots$ and so we have $\bigcup T=\bigcup\limits_{k=1}^{\ell}U_{n_k}=U_{n_\ell}=\overline{B}\big(a,\frac{1}{n_\ell}\big)^c$. Since a is a limit point of A we have $B\big(a,\frac{1}{n}\big)\cap A\neq\emptyset$, hence $\overline{B}\big(a,\frac{1}{n}\big)\cap A\neq\emptyset$, and so A is not a subset of $\bigcup T$. Since no finite subset of S covers A, it follows that A is not compact.

(b) Show, from the definition of compactness, that the set $B=\left\{\frac{n|n|}{1+n^2}\,\middle|\,n\in\mathbb{Z}\right\}\cup\{1,-1\}$ is compact.

Solution: Note that $\lim\limits_{n\to\infty}\dfrac{n|n|}{1+n^2}=1$ and $\lim\limits_{n\to-\infty}\dfrac{n|n|}{1+n^2}=-1$. Let S be any open cover of B. Since S covers B and $\pm1\in B$ we can choose $V,W\in S$ such that $1\in V$ and $-1\in W$. Since V and W are open we can choose $r>0$ such that $B(1,r)\subseteq V$ and $B(-1,r)\subseteq W$. Since $\lim\limits_{n\to\infty}\dfrac{n|n|}{1_+n^2}=1$ and $\lim\limits_{n\to\infty}\dfrac{n|n|}{1+n^2}=-1$ we can choose $N\in\mathbb{Z}^+$ such that for all $n\in\mathbb{Z}$, if $n\geq N$ then $\left|\frac{n|n|}{1+n^2}-1\right|<r$ so that $\frac{n|n|}{1+n^2}\in V$ and if $n\leq -N$ then $\left|\frac{n|n|}{1+n^2}+1\right|<r$ so that $\frac{n|n|}{1+n^2}\in W$. For each $n\in\mathbb{Z}$ with $-N<n<N$, choose $U_n\in S$ so that $\frac{n|n|}{1+n^2}\in U_n$. Then the set $T=\big\{U_n\big|-N<n<n\big\}\cup\{V,W\}$ is a finite subcover of S. Thus B is compact.

(c) Show that the set $O_n(\mathbb{R})=\big\{A\in M_n(\mathbb{R})\big|A^TA=I\big\}$ is compact. Here, we are identifying $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$, so that the dot product of two matrices is given by $A\bullet B=\sum\limits_{k,\ell}A_{k,\ell}B_{k,\ell}=\text{trace}(B^TA)$.

Solution: Note that for $A\in M_n(\mathbb{R})$ we have

$$A\in O_n(\mathbb{R})\iff A^TA=I\iff (A^TA)_{k,l}=I_{k,l}\text{ for all }k,l\iff\sum_{i=1}^{n}A_{i,k}A_{i,l}=\delta_{k,l}\text{ for all }k,l,$$

where

$$\delta_{k,l}=\begin{cases}1\text{ if }k=l\\0\text{ if }k\neq l.\end{cases}$$

For each pair k,l, define $f_{k,l}:M_n(\mathbb{R})\to\mathbb{R}$ by $f_{k,l}(A)=\sum\limits_{i=1}^{n}A_{i,k}A_{i,l}-\delta_{k,l}$. Note that each function $f_{k,l}$ is continuous since it is an elementary function on the n^2 variables $A_{i,j}$. We have

$$O_n(\mathbb{R})=\big\{A\in M_r(\mathbb{R})\big|f_{k,l}(A)=0\text{ for all }k,l\big\}=\bigcap_{k,l}\big\{A\in M_n(\mathbb{R})\big|f_{k,l}(A)=0\big\}=\bigcap_{k,l}f_{k,l}^{-1}(0).$$

Note that $f_{k,l}^{-1}(0)$ is the complement in $M_n(\mathbb{R})$ of the set $f_{k,l}^{-1}(\mathbb{R}\setminus\{0\})$. Since $\mathbb{R}\setminus\{0\}$ is open in $\mathbb{R}$ and each function $f_{k,l}$ is continuous, it follows that each set $f_{k,l}^{-1}(\mathbb{R}\setminus\{0\})$ is open, and hence each set $f_{k,l}^{-1}(0)$ is closed. Thus $O_n(\mathbb{R})$ is closed because it is the intersection of finitely many closed sets.

We claim that $O_n(\mathbb{R})$ is bounded. Let $A\in O_n(\mathbb{R})$. Let $u_1,u_2,\cdots,u_n$ be the columns of A. Note that

$$A^TA=\begin{pmatrix}{u_1}^T\\ \vdots\\ {u_n}^T\end{pmatrix}\big(u_1,\cdots,u_n\big)=\begin{pmatrix}u_1\bullet u_1 & u_1\bullet u_2 & \cdots & u_1\bullet u_n\\ \vdots & & & \vdots\\ u_n\bullet u_1 & u_n\bullet u_2 & \cdots & u_n\bullet u_n\end{pmatrix}$$

and so

$$\begin{aligned}A^TA=I&\implies(A^TA)_{k,k}=1\text{ for all }k\implies u_k\bullet u_k=1\text{ for all }k\implies|u_k|=1\text{ for all }k,l\\&\implies|A|^2=\sum_{k=1}^{n}\sum_{i=1}^{n}(A_{i,k})^2=\sum_{k=1}^{n}|u_k|^2=\sum_{k=1}^{n}1=n.\end{aligned}$$

Thus for every $A\in O_n(\mathbb{R}^n)$ we have $|A|=\sqrt{n}$ and so $O_n(\mathbb{R})$ is bounded, as claimed. We have shown that $O_n(\mathbb{R})$ is closed and bounded, and so it is compact, by the Heine Borel Theorm (which we can apply because we are identifying $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$).

7: For each of the following functions $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ or show that the limit does not exist.

(a) $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

Solution: Let $\theta \in \mathbb{R}$ and define $\alpha : \mathbb{R} \to \mathbb{R}^2$ by $\alpha(t) = (t\cos\theta, t\sin\theta)$. Then we have $\lim\limits_{t\to 0} \alpha(t) = (0,0)$ and $f(\alpha(t)) = \frac{t^2\cos^2\theta - t^2\sin^2\theta}{t^2\cos^2\theta + t^2\sin^2\theta} = \cos 2\theta$ for all $t \neq 0$, and so (by Composites and Limits) if $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ existed then it would be equal to $\cos 2\theta$. Since different choices of θ yield different values for the limit, the limit cannot exist.

(b) $f(x,y) = \dfrac{x^2 y^3}{x^4 + y^6}$

Solution: Consider the graph $z = f(x,y)$. The level set $y = c > 0$ is given by $z = g(x) = f(x,c) = \frac{c^3x^2}{x^4+c^6}$. Then

$$z' = g'(x) = \tfrac{c^3(2x(x^4+c^6)-(x^2)(4x^3))}{(x^4+c^6)^2} = \tfrac{c^3(2x)(c^6-x^4)}{(x^4+c^6)^2},$$

so we have $z' = 0$ when $x = 0$ and when $x = \pm c^{3/2}$. When $x = 0$ we have $z = 0$ and when $x = \pm c^{3/2}$ we have $z = \frac{c^3 \cdot c^3}{c^6 + c^6} = \frac{1}{2}$. The graph $z = f(x,y)$ with $y > 0$ has a maximum ridge of height $z = \frac{1}{2}$ along $x = \pm y^{3/2}$, that is $x^2 = y^3$.

Define $\alpha : \mathbb{R} \to \mathbb{R}^2$ by $\alpha(t) = (0,t)$. Then $\lim\limits_{t\to 0} \alpha(t) = (0,0)$ and $f(\alpha(t)) = 0$ for all $t \neq 0$, and so (by Composites and Limits) if $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ existed then it would be equal to 0. Define $\beta : \mathbb{R} \to \mathbb{R}^2$ by $\beta(t) = (t^3, t^2)$. Then $\lim\limits_{t\to 0} \beta(t) = (0,0)$ and $f(\beta(t)) = \frac{t^6 \cdot t^6}{t^{12} + t^{12}} = \frac{1}{2}$ for all $t \neq 0$, and so if $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ existed then it would be equal to $\frac{1}{2}$. Thus the limit cannot exist.

(c) $f(x,y) = \dfrac{x^4 y^5}{x^8 + y^6}$

Solution: Recall that for all $u, v \in \mathbb{R}$ we have $0 \leq (|u| - |v|)^2 = u^2 - 2|uv| + v^2$ and so $|uv| \leq \frac{1}{2}(u^2 + v^2)$. It follows that for all $(x,y) \neq (0,0)$ we have

$$\left|f(x,y) - 0\right| = \left|\tfrac{x^4y^5}{x^8+y^6}\right| = \tfrac{|x^4y^3|y^2}{x^8+y^6} \leq \tfrac{\frac{1}{2}(x^8+y^6)y^2}{x^8+y^6} = \tfrac{1}{2}\,y^2.$$

Given $\epsilon > 0$ choose $\delta = \sqrt{2\epsilon}$. Then for all x, y with $0 < \left|(x,y)\right| < \delta$ we have $0 < x^2 + y^2 < \delta^2$ and so

$$\left|f(x,y) - 0\right| \leq \tfrac{1}{2}\,y^2 \leq \tfrac{1}{2}(x^2 + y^2) < \tfrac{1}{2}\delta^2 = \epsilon.$$

8: Let $f : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^m$.

(a) Show that f is continuous if and only if $f^{-1}(F)$ is closed in A for every closed set F in B.

Solution: We already know that f is continuous if and only if $f^{-1}(E)$ is open in A for every open set E in B. Suppose that f is continuous. Let F be a closed set in B. Then $B \setminus F$ is open in B and so $f^{-1}(B \setminus F)$ is open in A and hence $A \setminus f^{-1}(B \setminus F)$ is closed in A. But notice that $f^{-1}(F) = A \setminus f^{-1}(B \setminus F)$ because for $a \in A$ we have

$$a \in f^{-1}(F) \iff f(a) \in F \iff f(a) \notin B \setminus F \iff a \notin f^{-1}(B \setminus F) \iff a \in A \setminus f^{-1}(B \setminus F).$$

Thus $f^{-1}(F)$ is closed in A for every closed set F in B.

Conversely, suppose that $f^{-1}(F)$ is closed in A for every closed set F in B. Let E be an open set in B. Then $B \setminus E$ is closed in B, hence $f^{-1}(B \setminus E)$ is closed in B, and so $A \setminus f^{-1}(B \setminus E)$ is open in A. But notice that $f^{-1}(E) = A \setminus f^{-1}(B \setminus E)$, as above. This shows that that $f^{-1}(E)$ is open in A for every open set E in B, and so f is continuous.

(b) Let E and F be closed sets in A with $E \cup F = A$. Let g be the restriction of f to E, and let h be the restriction of f to F. Show that f is continuous if and only if both g and h are continuous.

Solution: We begin by remarking that when $S \subseteq A \subseteq \mathbb{R}^n$, the open sets in S are the sets of the form $L \cap S$ with L being an open set in A. Indeed when L is open in A we can choose an open set U in $\mathbb{R}^n$ such that $L = U \cap A$, and then we have $L \cap S = (U \cap A) \cap S = U \cap S$ since $S \subseteq A$. On the other hand, when E is open in S we can choose an open set U in $\mathbb{R}^n$ such that $E = U \cap S$ and then the set $L = U \cap A$ is open in A with $L \cap S = (U \cap A) \cap S = U \cap S = E$. Similarly, the closed sets in S are the sets of the form $K \cap S$ with K being a closed set in A.

Suppose $f : A \to B$ is continuous. We claim that the restriction of f to *any* subset $S \subseteq A$ is continuous. Let $S \subseteq A$ and let $p : S \subseteq A \to B$ be the restriction of f to S. Let E be an open set in B. Then $f^{-1}(E)$ is open in A and so $S \cap f^{-1}(E)$ is open in S. But notice that $p^{-1}(E) = S \cap f^{-1}(E)$ since for $a \in A$ we have

$$\begin{aligned} a \in p^{-1}(E) &\iff a \in S \text{ and } p(a) \in E \iff a \in S \text{ and } f(a) \in E \\ &\iff a \in S \text{ and } a \in f^{-1}(E) \iff a \in S \cap f^{-1}(E). \end{aligned}$$

This shows that $p^{-1}(E)$ is open in S for every open set E in B, and so p is continuous in S.

Conversely, suppose that both of the two restrictions g and h are continuous. Let C be a closed set in B. Then $g^{-1}(C)$ is closed in E and $h^{-1}(C)$ is closed in F. Since $g^{-1}(C)$ is closed in E we can choose a closed set K in A so that $g^{-1}(C) = E \cap K$. Since E and K are both closed in A, it follows that $g^{-1}(C)$ is closed in A. Similarly, since $h^{-1}(C)$ is closed in F and F is closed in A, it follows that $h^{-1}(C)$ is closed in A. Since $g^{-1}(C)$ and $h^{-1}(C)$ are both closed in A, their union $g^{-1}(C) \cup h^{-1}(C)$ is closed in A. But notice that $f^{-1}(C) = g^{-1}(C) \cup h^{-1}(C)$ because for $a \in A$ we have

$$\begin{aligned} a \in f^{-1}(C) &\iff a \in A \text{ and } f(a) \in C \iff a \in E \cup F \text{ and } f(a) \in C \\ &\iff \big(a \in E \text{ and } f(a) \in C\big) \text{ or } \big(a \in F \text{ and } f(a) \in C\big) \\ &\iff \big(a \in E \text{ and } g(a) \in C\big) \text{ or } \big(a \in F \text{ and } h(a) \in C\big) \\ &\iff a \in g^{-1}(C) \text{ or } a \in h^{-1}(C). \end{aligned}$$

(c) Show that f is continuous if and only if for every $E \subseteq A$ we have $f\big(\overline{E}\,\big) \subseteq \overline{f(E)}$.

Solution: Suppose that f is continuous. Let $E \subseteq A$. Let $b \in f\big(\overline{E}\big)$, say $b = f(a)$ where $a \in A \cap \overline{E}$. We must show that $b \in \overline{f(E)}$. Let $r > 0$. Since $B_B(b,r)$ is open in B and f is continuous, $f^{-1}\big(B_B(b,r)\big)$ is open in A, so we can choose $s > 0$ so that $B_A(a,s) \subseteq f^{-1}\big(B_B(b,r)\big)$. Since $a \in A \cap \overline{E}$, we have $B_A(a,s) \cap E \neq \emptyset$, so we can choose a point $c \in B_A(a,s) \cap E$. Since $c \in B_A(a,s) \subseteq f^{-1}\big(B_B(b,r)\big)$ we have $f(c) \in B_B(b,r)$, and since $c \in E$ we have $f(c) \in f(E)$, and so $f(c) \in B_B(b,r) \cap f(E)$. Thus $B_B(b,r) \cap f(E) \neq \emptyset$ for all $r > 0$, so $b \in \overline{f(E)}$, as required.

Conversely, suppose that for every $E \subseteq A$ we have $f\big(\overline{E}\big) \subseteq \overline{f(E)}$. Let $K \subseteq B$ be closed in B. We claim that $f^{-1}(K)$ is closed in A. Let $C = f^{-1}(K)$. Note that $f(C) \subseteq K$. Let $x \in \overline{C}$. Then $f(x) \in f(\overline{C}) \subseteq \overline{f(C)} \subseteq \overline{K} = K$ and so $x \in f^{-1}(K) = C$. Thus $\overline{C} \subseteq C$. Of course we also have $C \subseteq \overline{C}$, so $C = \overline{C}$, and so C is closed, as claimed. Thus f is continuous.

9: (a) Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Show that if A is compact and f is continuous then f is uniformly continuous.

Solution: Suppose that A is compact and f is continuous. Let $\epsilon > 0$. For each $a \in A$, since f is continuous at a we can choose $\delta_a > 0$ such that $|x-a| < 2\delta_a \implies \big|f(x)-f(a)\big| < \frac{\epsilon}{2}$. Let $S = \big\{B(a,\delta_a)\big|a \in A\big\}$ and note that S is an open cover of A. Since A is compact, we can choose a finite subcover T of S, say $T = \big\{B(a_k,\delta_{a_k})\big|1 \leq k \leq \ell\big\}$. Let $\delta = \min\big\{\delta_{a_k}\big|1 \leq k \leq \ell\big\}$. Let $x, y \in A$ with $|x-y| < \delta$. Since T covers A we can choose an index k such that $x \in B(a_k,\delta_{a_k})$. Since $|x-a_k| < \delta_{a_k}$ and $|x-y| < \delta \leq \delta_{a_k}$ we have $|y-a_k| \leq 2\delta_{a_k}$. Since $|x-a_k| < 2\delta_{a_k}$ and $|y-a_k| < 2\delta_{a_k}$ we have $\big|f(x)-f(a_k)\big| < \frac{\epsilon}{2}$ and $\big|f(y)-f(a_k)\big| < \frac{\epsilon}{2}$ and hence $\big|f(x)-f(y)\big| < \epsilon$.

(b) Let $f : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^m$. Show that if A is compact and f is continuous and bijective then f^{-1} is continuous.

Solution: Suppose that A is compact and f is continuous and bijective, and let $g = f^{-1} : B \to A$. Let E be a closed set in A. By the Heine-Borel Theorem, A is closed and bounded. Since E is closed in A we can choose a closed set K in $\mathbb{R}^n$ such that $E = K \cap A$ (by Theorem 5.31). Since K and A are closed in $\mathbb{R}^n$, so is $E = K \cap A$ (by Theorem 5.14). Since $E \subseteq A \subseteq \mathbb{R}^n$ with E closed and A compact, it follows that E is compact (by Theorem 5.28). Since E is compact and f is continuous, it follows that $f(E)$ is compact (by Theorem 5.70 Part 2) hence $f(E)$ closed (by the Heine-Borel Theorem). Since f and g are inverses, we have $g^{-1}(E) = f(E)$, which is closed. Since $g^{-1}(E)$ is closed for every closed set E in A, it follows that g is continuous (by Theorem 5.69 Part 2, proved in Problem 8 (a)).

(c) Let $\emptyset \neq A, B \subseteq \mathbb{R}^n$. Define the **distance** between A and B to be

$$d(A,B) = \inf\big\{|x-y|\,\big|\,x \in A, y \in B\big\}.$$

Show that if A is compact and B is closed and $A \cap B = \emptyset$ then $d(A,B) > 0$.

Solution: Since B is closed, hence $B^c = \mathbb{R}^n \setminus B$ is open, for each $a \in A$ we can choose $r_a > 0$ so that $B(a, 2r_a) \subseteq B^c$. The set $S = \big\{B(a,r_a)\big|a \in A\big\}$ is an open cover of A. Since A is compact, we can choose a finite subcover $T \subseteq S$, say $T = \big\{B(a_1,r_{a_1}), B(a_2,r_{a_2}), \cdots, B(a_\ell,r_{a_\ell})\big\}$ where each $a_k \in A$. Let $r = \min\{r_{a_1}, r_{a_2}, \cdots, r_{a_\ell}\}$. We claim that $d(A,B) \geq r$. Let $x \in A$ and $y \in B$. Since T covers A, we can choose an index k so that $x \in B(a_k, r_{a_k})$ hence $|x-a_k| < r_{a_k}$. Since $y \in B$ and $B(a_k, 2r_{a_k}) \subseteq B^c$ we must have $|y-a_k| \geq 2r_{a_k}$. By the Triangle Inequality, $|y-a_k| \leq |y-x| + |x-a_k|$ hence $|y-x| \geq |y-a_k| - |x-a_k| \geq 2r_{a_k} - r_{a_k} = r_{a_k} \geq r$. Since $|y-x| \geq r$ for all $x \in A$ and $y \in B$ we have $d(A,B) = \inf\big\{|y-x|\,\big|\,x \in A, y \in B\big\} \geq r$, as claimed.

10: Let $A \subseteq \mathbb{R}^n$.

(a) For $a, b \in A$, write $a \sim b$ when there exists a continuous path in A from a to b. Show that $\sim$ is an equivalence relation on A (this means that for all $a, b, c \in A$ we have $a \sim a$, and if $a \sim b$ then $b \sim a$, and if $a \sim b$ and $b \sim c$ then $a \sim c$).

Solution: Let $a, b, c \in A$. We have $a \sim a$ because we can define $\alpha : [0, 1] \to A$ by $\alpha(t) = a$ for all t, and then α is continuous with $\alpha(0) = a$ and $\alpha(1) = a$, so α is a path in A from a to a.

Suppose that $a \sim b$. Let α be a path in A from a to b, so $\alpha : [0, 1] \to A$ is continuous with $\alpha(0) = a$ and $\alpha(1) = b$. Define $\beta : [0, 1] \to A$ by $\beta(t) = \alpha(1 - t)$. Note that β is continuous since it is the composite of the continuous map α with the continuous map $s : [0, 1] \to [0, 1]$ given by $s(t) = 1 - t$, and note that we have $\beta(0) = \alpha(1) = b$ and $\beta(1) = \alpha(0) = a$. Thus β is a path in A from b to a and so $b \sim a$.

Finally, suppose that $a \sim b$ and $b \sim c$. Let α be a path from a to b in A and let β be a path from b to c in A. Define $\gamma : [0, 1] \to A$ by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{, for } 0 \le t \le \frac{1}{2}, \\ \beta(2t - 1) & \text{, for } \frac{1}{2} \le t \le 1. \end{cases}$$

Note that $\gamma(0) = \alpha(0) = a$, $\gamma\left(\frac{1}{2}\right) = \alpha(1) = \beta(0) = b$, and $\gamma(1) = \beta(1) = c$. Gamma is continuous by Problem 8(b), because the sets $E = \left[0, \frac{1}{2}\right]$ and $F = \left[\frac{1}{2}, 1\right]$ are closed in $[0, 1]$ with $E \cup F = [0, 1]$, and the restriction of γ to E is given by $\alpha(2t)$, which is continuous (being the composite of two continuous functions), and the restriction of γ to F is given by $\beta(2t - 1)$, which is also continuous.

(b) Suppose that A is open and connected. Show that A is path connected.

Solution: The empty set is open, connected and path-connected (vacuously). Suppose $A \neq \emptyset$ and let $a \in A$. Let

$$E = \left\{ b \in A \middle| a \sim b \right\}.$$

We claim that E is open in A. Let $b \in E$. Since $b \in A$ and A is open in $\mathbb{R}^n$, we can choose $r > 0$ so that $B(b, r) \subseteq A$. Let $c \in B(b, r)$. Since $b \in E$ we have $a \sim b$. Since $c \in B(b, r) \subseteq A$ we have $b \sim c$, indeed we can define $\alpha : [0, 1] \to B(b, r) \subseteq A$ by $\alpha(t) = b + t(c - b)$ and then α is continuous (since it elementary), and $\alpha(0) = b$ and $\alpha(1) = c$, and $\alpha(t) \in B(b, r)$ for all $t \in [0, 1]$ because $\big|\alpha(t) - b\big| = \big|t(c - b)\big| = |t||c - b| \le |c - b| < r$. Since $a \sim b$ and $b \sim c$ we have $a \sim c$ by Part (a). Since $a \sim c$ we have $c \in E$, hence $B(b, r) \subseteq E$. This shows that E is open in $\mathbb{R}^n$ hence also in A.

We claim that E is also closed in A. Let $b \in A \setminus E$. Since $b \in A$ and A is open in $\mathbb{R}^n$, we can choose $r > 0$ so that $B(b, r) \subseteq A$. Let $c \in B(b, r)$. Since $b \notin E$ we have $a \not\sim b$. Since $c \in B(b, r) \subseteq A$ we have $b \sim c$, as above. It follows from Part (a) that $a \not\sim c$ since otherwise we would have $a \sim c$ and $c \sim b$ and hence $a \sim b$. Since $c \not\sim a$ we have $c \in A \setminus E$. Thus $B(b, r) \subseteq A \setminus E$. This shows that $A \setminus E$ is open (both in $\mathbb{R}^n$ and in A) so that E is closed in A.

Since A is connected, the only subsets of A which are both open and closed are $\emptyset$ and A. Since E is both open and closed we must have $E = \emptyset$ or $E = A$. Since $a \sim a$ we have $a \in E$ so $E \neq \emptyset$ and so $E = A$. Since $A = E = \left\{ b \in A \middle| a \sim b \right\}$ we have $a \sim b$ for every $b \in A$. Thus A is path connected.