PMATH 333 Intro to Real Analysis, Solutions to the Midterm Test, Fall 2024

[10] **1:** (a) Let R be a ring. Using Rules R0-R7 (with only one rule used at each step), prove that for all $a \in R$, if $1 + a = 0$ then $a \cdot a = 1$.

Solution: Let $a \in R$. Suppose that $1 + a = 0$. Then

$$\begin{aligned} a \cdot a &= a \cdot a + 0 \ \text{, by R3} \\ &= 0 + a \cdot a \ \text{, by R2} \\ &= (1 + a) + a \cdot a \ \text{, since } 1 + a = 0 \\ &= 1 + (a + a \cdot a) \ \text{, by R1} \\ &= 1 + (a \cdot 1 + a \cdot a) \ \text{, since } a = a \cdot 1 \text{ by R6} \\ &= 1 + a \cdot (1 + a) \ \text{, by R7} \\ &= 1 + a \cdot 0 \ \text{, since } 1 + a = 0 \\ &= 1 + 0 \ \text{, since } a \cdot 0 = 0 \text{ by R0} \\ &= 1 \ \text{, by R3.} \end{aligned}$$

(b) Let F be an ordered field. Using the result of Part (a), together with Rules R0-R9 and O1-O5 (with only one rule used at each step), prove that $0 \le 1$.

Solution: By O1, we know that either $0 \le 1$ or $1 \le 0$. Suppose that $1 \le 0$. By R3, we can choose $a \in F$ such that $1 + a = 0$. By Part (a), we know that $a \cdot a = 1$. We have

$$\begin{aligned} 1 &\le 0 \ \text{, by assumption} \\ 1 + a &\le 0 + a \ \text{, by O3} \\ 0 &\le 0 + a \ \text{, since } 1 + a = 0 \\ 0 &\le a + 0 \ \text{, by R2} \\ 0 &\le a \ \text{, by R3} \\ 0 &\le a \cdot a \ \text{, by O5 (taking } b = a) \\ 0 &\le 1 \ \text{, since } a \cdot a = 1 \text{ by Part (a).} \end{aligned}$$

(c) Determine whether the following formula is true, when the variables represent sets:

$$\exists w \ \forall v \ \big(v \in w \leftrightarrow \exists u \, \forall y \big(y \in v \leftrightarrow \forall x \, (x \in y \to x \in u)\big)\big).$$

Solution: Recall that for a set u, the power set of u is the set $P(u) = \big\{y \,\big|\, y \subseteq u\big\}$. Let us say that "$v$ is a power set" when $v = P(u)$ for some set u. In the class of sets, we have

$$\begin{aligned} \forall x \, (x \in y \to x \in u) &\iff y \subseteq u \\ \forall y \big(y \in v \leftrightarrow \forall x \, (x \in y \to x \in u)\big) &\iff v = P(u) \\ \exists u \, \forall y \big(y \in v \leftrightarrow \forall x \, (x \in y \to x \in u)\big) &\iff v \text{ is a power set} \\ \forall v \, \big(v \in w \leftrightarrow \exists u \, \forall y \big(y \in v \leftrightarrow \forall x \, (x \in y \to x \in u)\big)\big) &\iff w = \big\{v \,\big|\, v \text{ is a power set}\big\} \end{aligned}$$

so the given formula states that the class $w = \big\{v \,\big|\, v \text{ is a power set}\big\} = \big\{v \,\big|\, \exists u \ v = P(u)\big\}$ is a set. We claim that this is FALSE. If w was a set, then $\bigcup w$ would also be a set (by the Union Axiom). But then for every set u we would have $u \in P(u) \subseteq \bigcup w$, so that $\bigcup w$ is the class of all sets (which, as we know, is not a set).

[10] **2:** (a) Let $x_n = \frac{(2n+1)^2}{n^2+1}$ for $n \in \mathbb{Z}^+$. Use the definition of the limit to prove that $\lim_{n\to\infty} x_n = 4$.

Solution: For all $n \in \mathbb{Z}^+$ we have

$$\left|x_n - 4\right| = \left|\frac{(2n+1)^2}{n^2+1} - 4\right| = \left|\frac{(2n+1)^2-4(n^2+1)}{n^2+1}\right| = \left|\frac{4n-3}{n^2+1}\right| = \frac{4n-3}{n^2+1} < \frac{4n}{n^2} = \frac{4}{n}.$$

Let $\epsilon > 0$. Choose $N \in \mathbb{Z}^+$ such that $\frac{4}{n} < \epsilon$. Then for all $n \in \mathbb{Z}^+$ with $n \geq N$ we have $\left|x_n - 4\right| < \frac{4}{n} \leq \frac{4}{N} < \epsilon$.

(b) Prove that every Cauchy sequence in $\mathbb{R}$ converges in $\mathbb{R}$.

Solution: Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in $\mathbb{R}$. We claim that $(x_n)_{n\geq 1}$ is bounded. Since $(x_n)_{n\geq 1}$ is Cauchy, we can choose $N \in \mathbb{Z}^+$ such that $n, m \geq N \Longrightarrow |x_n - x_m| < 1$. Let $m = N$. Then for all $n \geq N$ we have $|x_n - x_m| < 1$ so that $x_m - 1 < x_n < x_m + 1$. It follows that $(x_n)_{n\geq 1}$ is bounded above by $\max\left\{x_1, x_2, \cdots, x_{m-1}, x_m + 1\right\}$ and bounded below by $\min\left\{x_1, x_2, \cdots, x_{m-1}, x_m - 1\right\}$.

Because $(x_n)_{n\geq 1}$ is bounded, it has a convergent subsequence, by the Bolzano-Weierstrass Theorem. Let $(x_{n_k})_{k\geq 1}$ be a convergent subsequence of $(x_n)_{n\geq 1}$ and let $a = \lim_{k\to\infty} x_{n_k}$. We claim that $\lim_{n\to\infty} x_n = a$. Let $\epsilon > 0$. Since (x_n) is Cauchy, we can choose $N \in \mathbb{Z}^+$ so that $n, m \geq N \Longrightarrow |x_n - x_m| < \frac{\epsilon}{2}$. Since $x_{n_k} \to a$ we can choose $K \in \mathbb{Z}^+$ so that $k \geq K \Longrightarrow |x_{n_k} - a| < \frac{\epsilon}{2}$. Since $n_k \to \infty$, we can choose an index $k \geq K$ so that $n_k \geq N$. Then for all $n \geq N$ we have $|x_n - a| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus $x_n \to a$, as claimed.

[10] **3:** (a) Let $f : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$ with $\lim_{n\to\infty} f(x) = a \in \mathbb{R}$. Prove that f is uniformly continuous.

Solution: Let $\epsilon > 0$. Since $\lim_{n\to\infty} f(x) = a$, we can choose $r > 0$ such that $x \geq r \Longrightarrow |f(x) - a| < \frac{\epsilon}{2}$. Since f is continuous on $[0, r + 1]$, it is also uniformly continuous on $[0, r + 1]$, so we can choose $\delta_0 > 0$ such that for all $x, y \in [0, r + 1]$, we have $|x - y| < \delta_0 \Longrightarrow |f(x) - f(y)| < \epsilon$. Let $\delta = \min(\delta_0, 1)$. Let $x, y \in [0, \infty)$ with $|x - y| < \delta$ so that $|x - y| < \delta_0$ and $|x - y| < 1$. Since $|x - y| < 1$, either we have $x, y \in [0, r + 1]$ or we have $x, y \in [r, \infty)$. In the case that $x, y \in [0, r + 1]$, since $|x - y| < \delta_0$ we have $|f(x) - f(y)| < \epsilon$ (by the choice of δ_0). In the case that $x, y \in [r, \infty)$ we have $|f(x) - a| < \frac{\epsilon}{2}$ and $|f(y) - a| < \frac{\epsilon}{2}$ so that $|f(x) - f(y)| \leq |f(x) - a| + |a - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

(b) Prove that every continuous function $f : [a, b] \to \mathbb{R}$ attains its maximum value.

Solution: First we claim that f is bounded above. Suppose, for a contradiction, that f is not bounded above. For each $n \in \mathbb{Z}^+$, choose $x_n \in [a, b]$ such that $f(x_n) \geq n$. By the Bolzano-Weierstrass Theorem, we can choose a convergent subsequence $(x_{n_k})_{k\geq 1}$. Let $r = \lim_{k\to\infty} x_{n_k}$. Since $a \leq x_{n_k} \leq b$ for all k, it follows that $a \leq r \leq b$ by Comparison, so we have $r \in [a, b]$. Since $f(x_{n_k}) \geq n_k$ and $n_k \to \infty$ we have $f(x_{n_k}) \to \infty$ as $k \to \infty$, by Comparison. But by the Sequential Characterization of Continuity, we also have $f(x_{n_k}) \to f(r) \in \mathbb{R}$, which gives the desired contradiction. Thus f is bounded above, as claimed.

Since the range $f([a, b])$ is nonempty and bounded above, it has a supremum. Let $m = \sup f([a, b])$. By the Approximation Property of the supremum, for each $n \in \mathbb{Z}^+$ we can choose $y_n \in [a, b]$ such that $m - \frac{1}{n} \leq f(y_n) \leq m$. By the Bolzano-Weierstrass Theorem, we can choose a convergent subsequence $(y_{n_k})_{k\geq 1}$. Let $c = \lim_{k\to\infty} y_{n_k}$. Since we have $m - \frac{1}{n_k} \leq f(y_{n_k}) \leq m$ and $\frac{1}{n_k} \to 0$, it follows that $f(y_{n_k}) \to m$ as $k \to \infty$ by the Squeeze Theorem. Since f is continuous at c, by the Sequential Characterization of Continuity we have $f(y_{n_k}) \to f(c)$, and so by the Uniqueness of Limits, we have $f(c) = m$. Thus f attains its maximum value at c.

[10] **4:** (a) Prove, from the definition of the integral, that the function $f:[1,3]\to\mathbb{R}$ given by $f(x)=2x-1$ is integrable on $[1,3]$ with $\int_1^3 f(x)\,dx=6$.

Solution: Let $\epsilon>0$. Choose $\delta=\frac{\epsilon}{4}$. Let X be any partition with $|X|<\delta$. Let $t_k\in[x_{k-1},x_k]$ and let $S=\sum\limits_{k=1}^n f(t_k)\Delta_k x$. Choose $s_k=\frac{x_k+x_{k-1}}{2}$. Since $s_k,t_k\in[x_{k-1},x_k]$ we have $|t_k-s_k|\le\Delta_k x\le|X|<\delta$. Note that

$$\begin{aligned}\sum_{k=1}^n f(s_k)\Delta_k x &= \sum_{k=1}^n (2s_k-1)\Delta_k x=\sum_{k=1}^n (x_k+x_{k-1}-1)(x_k-x_{k-1})\\ &=\sum_{k=1}^n \big((x_k+x_{k-1})(x_k-x_{k-1})-(x_k-x_{k-1})\big)=\sum_{k=1}^n ({x_k}^2-x_{k-1}^2)-\sum_{k=1}^n (x_k-x_{k-1})\\ &=({x_n}^2-{x_0}^2)-(x_n-x_0)=(3^2-1^2)-(3-1)=6.\end{aligned}$$

and so

$$\begin{aligned}\big|S-6\big| &= \Big|\sum_{k=1}^n f(t_k)\Delta_k x-\sum_{k=1}^n f(s_k)\Delta_k x\Big|=\Big|\sum_{k=1}^n \big(f(t_k)-f(s_k)\big)\Delta_k x\Big|=\Big|\sum_{k=1}^n \big((2t_k-1)-(2s_k-1)\big)\Delta_k x\Big|\\ &=\Big|\sum_{k=1}^n 2(t_k-s_k)\Delta_k x\Big|\le\sum_{k=1}^n 2|t_k-s_k|\Delta_k x<\sum_{k=1}^n 2\delta\Delta_k x=2\delta\sum_{k=1}^n \Delta_k x=2\delta\cdot 2=\epsilon.\end{aligned}$$

(b) Fix $\ell\in\mathbb{Z}^+$. Define $f:[0,1]\to\mathbb{R}$ by $f\big(\frac{r}{\ell}\big)=r$ for each $r\in\{0,1,2,\cdots,\ell\}$, and $f(x)=0$ when $x\neq\frac{r}{\ell}$ for any such r. Prove, from the definition, that f is integrable on $[0,1]$.

Solution: Let $I=0$. Let $\epsilon>0$. Choose $\delta=\min\big\{\frac{1}{\ell},\frac{\epsilon}{2\ell^2}\big\}$. Let $X=\{x_0,x_1,\cdots,x_n\}$ be a partition of $[0,1]$ with $|X|<\delta$. Let $I_k=[x_{k-1},x_k]$, let $t_k\in I_k$, and let $S=\sum\limits_{k=1}^n f(t_k)\Delta_k x$. Since $|X|<\delta\le\frac{1}{\ell}$, each interval I_k contains at most one of the points $\frac{r}{\ell}$. Also note that 0 and 1 each lie in exactly 1 of the intervals I_k (namely $0\in I_0$ and $1\in I_n$) while for $0<r<\ell$, $\frac{r}{\ell}$ lies in at most 2 of the intervals I_k, and so the number of indices k for which I_k contains one of the points $\frac{r}{\ell}$ is at most 2ℓ. Let K be the set of indices k for which I_k contains one of the points $\frac{r}{\ell}$, and let $\#K$ be the number of elements in K, so $\#K\le 2\ell$. When $k\notin K$ we have $f(x)=0$ for all $x\in I_k$ so that, in particular, $f(t_k)=0$. When $k\in K$ with $\frac{r}{\ell}\in I_k$, we have $f(x)=0$ for $x\neq\frac{r}{\ell}$ and $f\big(\frac{r}{\ell}\big)=r\le\ell$, so that $0\le f(t_k)\le r\le\ell$. Thus

$$0\le\sum_{k=1}^n f(t_k)\Delta_k x=\sum_{k\in K} f(t_k)\Delta_k x\le\sum_{k\in K}\ell\,\Delta_k x<\sum_{k\in K}\ell\delta=\ell\delta\,\#K\le\ell\delta\cdot 2\ell\le\epsilon$$

so that $|S-I|<\epsilon$, as required.