PMATH 336 Intro to Group Theory, Solutions to Assignment 3

1: Let $\alpha, \beta \in S_8$ be given by the following table of values:

k	1	2	3	4	5	6	7	8
$\alpha(k)$	4	6	3	5	7	8	1	2
$\beta(k)$	2	6	7	4	8	3	1	5

For each of the following permutations, write the permutation as a product of disjoint cycles and determine its order and its parity.

(a) α (b) β (c) $\alpha\beta$ (d) $\left(\alpha\beta^{-1}\right)^{20}$

Solution: Writing $\gamma = (\alpha\beta^{-1})^{20}$, we have

$$\begin{aligned}
\alpha &= (1,4,5,7)(2,6,8)\ ,\ |\alpha| = 12\ ,\ (-1)^\alpha = -1\,,\\
\beta &= (1,2,6,3,7)(5,8)\ ,\ |\beta| = 10\ ,\ (-1)^\beta = -1\,,\\
\alpha\beta &= (1,6,3)(2,8,7,4,5)\ ,\ |\alpha\beta| = 15\ ,\ (-1)^{\alpha\beta} = 1\,,\\
\beta^{-1} &= (1,7,3,6,2)(5,8)\ ,\ \alpha\beta^{-1} = (2,4,5)(3,8,7)\ ,\ |\alpha\beta^{-1}| = 3\,,\\
\gamma &= (\alpha\beta^{-1})^{20} = (\alpha\beta^{-1})^2 = (2,5,4)(3,7,8)\ ,\ |\gamma| = 3 \text{ and } (-1)^\gamma = 1.
\end{aligned}$$

2: (a) Find the maximum of the orders of the elements in S_8.

Solution: An 8-cycle has order 8. A 7-cycle has order 7. A 6-cycle, or the product of a 6-cycle with a 2-cycle has order 6. A 5-cycle has order 5, the product of a 5-cycle with a 2-cycle has order 10 and the product of a 5-cycle with a 3-cycle has order 15. All other permutations in S_8 contain cycles of lengths 1, 2, 3 and 4 and their order is at most $\text{lcm}(1,2,3,4) = 12$. Thus the maximum of the orders of the elements in S_8 is 15.

(b) Find the number of elements of order 6 in S_8.

Solution: The following table shows the possible forms for the elements of order 6:

form	no. of elements	parity
$(abc)(de)$	$\binom{8}{3}\cdot 2\cdot\binom{5}{2} = 1120$	$-$
$(abc)(de)(fg)$	$\binom{8}{3}\cdot 2\cdot\binom{5}{4}\cdot 3 = 1680$	$+$
$(abc)(def)(gh)$	$\binom{8}{6}\cdot 5\cdot 4\cdot 2 = 1120$	$-$
$(abcdef)$	$\binom{8}{6}\cdot 5! = 3360$	$-$
$(abcdef)(gh)$	$\binom{8}{6}\cdot 5! = 3360$	$+$

Altogether, there are 10640 elements of order 6 in S_8.

(c) Find the number of cyclic subgroups of order 6 in A_8.

Solution: From the above table, there are 5040 elements of order 6 in A_8. So (by Corollary 2.28) the number of cyclic subgroup of order 6 in A_8 is $5040/\varphi(6) = 5040/2 = 2520$.

3: Let $\alpha = (1234)(5678)$ and $\beta = (123)(456)$ in S_8.

(a) Express α as a product of 2-cycles and as a product of 3-cycles.

Solution: There are many ways to do this. For example $\alpha = (1234)(5678) = (14)(13)(12)(58)(57)(56) = (134)(12)(58)(567) = (134)(185)(125)(567)$.

(b) Find $|Cl(\beta)|$, that is find the number of elements in the conjugacy class of β.

Solution: The elements in S_8 which lie in the conjugacy class of β are the elements of the form $(abc)(def)$ with a, b, c, d, e, f distinct. The number of such elements is $\binom{8}{6} \cdot 5 \cdot 4 \cdot 2 = 1120$.

(c) Find all the elements $\sigma \in S_8$ such that $\sigma^2 = \beta$.

Solution: We have $\alpha^6 = \beta^3 = (1)$, so $|\alpha| = 1, 2, 3$ or 6. We cannot have $|\alpha| = 1$ since $\alpha \neq (1)$ (otherwise $\alpha^2 = (1) \neq \beta$), and we cannot have $|\alpha| = 2$ since $\alpha^2 = \beta \neq (1)$. Thus $|\alpha| = 3$ or 6. Case 1: if $|\alpha| = 3$ then α is of the form (abc) or the form $(abc)(def)$. If $\alpha = (abc)$ then $\alpha^2 = (acb) \neq \beta$. If $\alpha = (abc)(def)$ then $\alpha^2 = (acb)(dfe)$, and so $\alpha^2 = \beta \iff \alpha = (132)(465)$. Case 2: if $|\alpha| = 6$ then α is of one of the forms listed in the above table. If $\alpha = (abc)(de)$ or $(abc)(de)(fg)$ then $\alpha^2 = (acb) \neq \beta$. If $\alpha = (abc)(def)(gh)$ then $\alpha^2 = (acb)(dfe)$ and so $\alpha^2 = \beta \iff \alpha = (132)(465)(78)$. If $\alpha = (abcdef)$ or $(abcdef)(gh)$ then $\alpha^2 = (ace)(bdf)$, so we have $\alpha^2 = \beta \iff \alpha = (142536)$, (152634) or (162435), or $\alpha = (142536)(78)$, $(152634)(78)$ or $(162435)(78)$. Thus there are 8 elements $\alpha \in S_8$ with $\alpha^2 = \beta$, namely

$$\alpha \in \big\{(132)(465), (132)(465)(78), (142536), (152634), (162435), (142536)(78), (152634)(78), (162435)(78)\big\}.$$

4: (a) Find the number of elements of each order in $A_4 \times D_4$.

Solution: The number of elements of each order in S_4, A_4, D_4 and $A_4 \times D_4$ are given in the following tables:

In S_4

form of α	$\lvert\alpha\rvert$	$(-1)^\alpha$	# of α
$(abcd)$	4	-1	6
(abc)	3	1	8
$(ab)(cd)$	2	1	3
(ab)	2	-1	6
(a)	1	1	1

In A_4

$\lvert\alpha\rvert$	# of α
1	1
2	3
3	8

In D_4

$\lvert X\rvert$	X	# of X
1	I	1
2	R_2, F_0, F_1, F_2, F_3	5
4	R_1, R_3	2

In $A_4 \times D_4$

$\lvert\alpha\rvert$	# of α	$\lvert X\rvert$	# of X	$\lvert(\alpha, X)\rvert$	# of (α, X)
1	1	1	1	1	1
1	1	2	5	2	5
1	1	4	2	4	2
2	3	1	1	2	3
2	3	2	5	2	15
2	3	4	2	4	6
3	8	1	1	3	8
3	8	2	5	6	40
3	8	4	2	12	16

In $A_4 \times D_4$, Summary

$\lvert(\alpha, X)\rvert$	3 of (α, X)
1	1
2	23
3	8
4	8
6	40
12	16

(b) Find the number of elements of each order in $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$.

Solution: The number of elements of each order in $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$ are given in the following tables:

In $\mathbb{Z}_2 \times \mathbb{Z}_4$

$\lvert a\rvert$	# of a	$\lvert b\rvert$	# of b	$\lvert(a, b)\rvert$	# of (a, b)
1	1	1	1	1	1
1	1	2	1	2	1
1	1	4	2	4	2
2	1	1	1	2	1
2	1	2	1	2	1
2	1	4	2	4	2

In $\mathbb{Z}_2 \times \mathbb{Z}_4$, Summary

$\lvert(a, b)\rvert$	# of (a, b)
1	1
2	3
4	4

In $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$

$\lvert(a, b)\rvert$	# of (a, b)	$\lvert c\rvert$	# of c	$\lvert(a, b, c)\rvert$	# of (a, b, c)
1	1	1	1	1	1
1	1	2	1	2	1
1	1	3	2	3	2
1	1	6	2	6	2
2	3	1	1	2	3
2	3	2	1	2	3
2	3	3	2	6	6
2	3	6	2	6	6
4	4	1	1	4	4
4	4	2	1	4	4
4	4	3	2	12	8
4	4	6	2	12	8

In $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$, Summary

$\lvert(a, b, c)\rvert$	# of (a, b, c)
1	1
2	7
3	2
4	8
6	14
12	16