Chapter 7. Isometries and Symmetry Groups

7.1 Definition: For a map $S : \mathbb{R}^n \to \mathbb{R}^n$, we say that S **preserves distance** when

$$\big\|S(x) - S(y)\big\| = \|x - y\|$$

for all $x, y \in \mathbb{R}^n$. An **isometry** on $\mathbb{R}^n$ is an invertible map $S : \mathbb{R}^n \to \mathbb{R}^n$ which preserves distance. The set of all isometries on $\mathbb{R}^n$ is denoted by $\text{Isom}(\mathbb{R}^n)$.

7.2 Theorem: *The set of isometries on $\mathbb{R}^n$ is a group under composition.*

Proof: The identity map $I : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry because $\big\|I(x) - I(y)\big\| = \|x - y\|$ for all $x, y \in \mathbb{R}^n$. Note that if $S, T \in \text{Isom}(\mathbb{R}^n)$ then we have $ST \in \text{Isom}(\mathbb{R}^n)$ because for $x, y \in \mathbb{R}^n$ we have

$$\big\|S(T(x)) - S(T(y))\big\| = \big\|T(x) - T(y)\big\| = \|x - y\|.$$

Finally, note that if $S \in \text{Isom}(\mathbb{R}^n)$ then $S^{-1} \in \text{Isom}(\mathbb{R}^n)$ because given $u, v \in \mathbb{R}^n$, if we let $x = S^{-1}(u)$ and $y = S^{-1}(v)$ so that $u = S(x)$ and $v = S(y)$ then we have

$$\big\|S^{-1}(u) - S^{-1}(v)\big\| = \|x - y\| = \|S(x) - S(y)\| = \|u - v\|.$$

Thus $\text{Isom}(\mathbb{R}^n) \leq \text{Perm}(\mathbb{R}^n)$ by the Subgroup Test.

7.3 Example: For a vector $u \in \mathbb{R}^n$, the **translation** by u is the map $T_u : \mathbb{R}^n \to \mathbb{R}^n$ given by $T_u(x) = x + u$. Note that T_u is an isometry on $\mathbb{R}^n$ because

$$\big\|T_u(x) - T_u(y)\big\| = \big\|(u + x) - (u + y)\big\| = \|x - y\|\,.$$

7.4 Example: If $A \in O_n(\mathbb{R})$, so that $A^T A = I$, then the map $S : \mathbb{R}^n \to \mathbb{R}^n$ given by $S(x) = Ax$ is an isometry because for $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned}\big\|Ax - Ay\big\|^2 = \big\|A(x - y)\big\|^2 &= \big(A(x - y)\big)^T\big(A(x - y)\big)\\ &= (x - y)^T A^T A(x - y) = (x - y)^T(x - y) = \|x - y\|^2.\end{aligned}$$

7.5 Example: For a vector space U in $\mathbb{R}^n$, the **reflection** in U is the map $F_U : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$F_U(x) = x - 2\,\text{Proj}_{U^\perp}(x)$$

where $\text{Proj}_{U^\perp}(x)$ is the orthogonal projection of x onto $U^\perp$. When $\{u_1, u_2, \cdots, u_k\}$ is an orthonormal basis for $U^\perp$ and $A = \big(u_1, u_2, \cdots, u_k\big) \in M_{n\times k}(\mathbb{R})$, recall that

$$\begin{aligned}\text{Proj}_{U^\perp}(x) &= \sum_{i=1}^{n}(x \cdot u_i)u_i = AA^T x\,,\\ F_U(x) &= x - 2\,AA^T x = (I - 2AA^T)x.\end{aligned}$$

Note that since $\{u_1, u_2, \cdots, u_k\}$ is orthonormal, we have $A \in O_n(\mathbb{R})$, that is $A^T A = I$, and it follows that $(I - 2AA^T) \in O_n(\mathbb{R})$ because

$$\begin{aligned}(I - 2AA^T)^T(I - 2AA^T) &= I - 2AA^T - 2AA^T + 4AA^T AA^T\\ &= I - 4AA^T + 4A(A^T A)A^T = I - 4AA^T + 4AA^T = I.\end{aligned}$$

This shows that $F_U \in O_n(\mathbb{R})$ and hence $F_U \in \text{Isom}(\mathbb{R}^n)$. In particular, when U is a hyperspace (that is a vector space of dimension $n - 1$) and u is a non-zero vector in $U^\perp$, we have

$$\text{Proj}_{U^\perp}(x) = \text{Proj}_u(x) = \frac{x \cdot u}{\|u\|^2}\,u \quad\text{and}\quad F_U(x) = x - 2\frac{x \cdot u}{\|u\|^2}\,u.$$

7.6 Example: An **affine space** in $\mathbb{R}^n$ is a set of the form $P = p + U = \{p + x \,|\, x \in U\}$ for some point $p \in \mathbb{R}^n$ and some vector space $U \in \mathbb{R}^n$. For an affine space $P = p + U$ in $\mathbb{R}^n$, the **reflection** in P is the map $F_P : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$F_P(x) = p + F_U(x - p).$$

Note that $F_P \in \mathrm{Isom}(\mathbb{R}^n)$ because F_P is equal to the composite $F_P = T_p F_U T_{-p}$.

7.7 Theorem: *(The Algebraic Classification of Isometries) A map $S : \mathbb{R}^n \to \mathbb{R}^n$ preserves distance if and only if S is of the form $S(x) = Ax + b$ for some $A \in O_n(\mathbb{R})$ and some $b \in \mathbb{R}^n$.*

Proof: First note that if $S(x) = Ax + b$ where $A \in O_n(\mathbb{R})$ and $b \in \mathbb{R}^n$, then S is the composite $S = T_b A$, which is an isometry.

Conversely, suppose that S is an isometry. Let $b = S(0)$ and define $L : \mathbb{R} \to \mathbb{R}$ by $L(x) = S(x) - b$. Note that $S(0) = 0$ and that for $x \in \mathbb{R}^n$ we have

$$\|L(x)\| = \|L(x) - L(0)\| = \|(S(x) - b) - (S(0) - b)\| = \|S(x) - S(0)\| = \|x - 0\| = \|x\|.$$

For $x, y \in \mathbb{R}^n$, we have

$$\|x - y\|^2 = (x - y) \cdot (x - y) = x \cdot x - x \cdot y - y \cdot x + y \cdot y = \|x\|^2 - 2\, x \cdot y + \|y\|^2$$

from which we obtain the **Polarization Identity**:

$$x \cdot y = \tfrac{1}{2}\big(\|x\|^2 + \|y\|^2 - \|x - y\|^2\big).$$

For $x, y \in \mathbb{R}^n$, using the Polarization Identity, we have

$$L(x) \cdot L(y) = \tfrac{1}{2}\big(\|L(x)\|^2 + \|L(y)\|^2 - \|L(x) - L(y)\|^2\big) = \tfrac{1}{2}\big(\|x\|^2 + \|y\|^2 - \|x - y\|^2\big) = \|x - y\|^2.$$

In particular, $L(e_i) \cdot L(e_j) = e_i \cdot e_j = \delta_{i,j}$ for all i, j, so the set $\{L(e_1), L(e_2), \cdots, L(e_n)\}$ is an orthonormal basis for $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, if we write $x = \sum_{i-1}^{n} x_i e_i$ and $L(x) = \sum_{i=1}^{n} t_i L(e_i)$ then we have

$$t_k = L(x) \cdot L(e_k) = x \cdot e_k = x_k$$

and so we have $L(x) = \sum x_k L(e_k) = Ax$ where $A = \big(L(e_1), L(e_2), \cdots, L(e_n)\big) \in M_n(\mathbb{R})$. Since $\{L(e_1), L(e_2), \cdots, L(e_n)\}$ is an orthonormal set, it follows that $A^T A = I$ so we have $A \in O_n(\mathbb{R})$. Thus $S(x) = Ax + b$ with $A \in O_n(\mathbb{R})$ and $b \in \mathbb{R}^n$, as required.

7.8 Corollary: *Every distance preserving map $S : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry.*

Proof: If $S : \mathbb{R}^n \to \mathbb{R}^n$ preserves distance then S is invertible; indeed if S is given by $S(x) = Ax + b$ with $A \in O_n(\mathbb{R})$ and $b \in \mathbb{R}^n$ then S^{-1} is given by $S^{-1}(x) = A^{-1}x - A^{-1}b$.

7.9 Definition: Let $S \in \mathrm{Isom}(\mathbb{R}^n)$, say $S(x) = Ax + b$ with $A \in O_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. Note that since $A^T A = I$ we have $\det(A) = \pm 1$. We say that S **preserves orientation** when $\det(A) = 1$, and we say that S **reverses orientation** when $\det(A) = -1$. We write

$$\mathrm{Isom}_+(\mathbb{R}^n) = \{S \in \mathrm{Isom}(\mathbb{R}^n) \,|\, S \text{ preserves orientation}\}.$$

7.10 Definition: For a nonempty set $X \subseteq \mathbb{R}^n$, the **symmetry group** of X in $\mathbb{R}^n$ and the **rotation group** of X in $\mathbb{R}^n$ are the groups

$$\mathrm{Sym}(X) = \{S \in \mathrm{Isom}(\mathbb{R}^n) \,|\, S(x) \in X \text{ for all } x \in X\} = \{S \in \mathrm{Isom}(\mathbb{R}^n) \,|\, S(X) = X\}$$
$$\mathrm{Rot}(X) = \mathrm{Sym}(X) \cap \mathrm{Isom}_+(\mathbb{R}^n) = \{S \in \mathrm{Isom}_+(\mathbb{R}^n) \,|\, S(X) = X\}.$$

7.11 Theorem: (*The Fixed Point Theorem*) *Let G be a finite subgroup of* $\text{Isom}(\mathbb{R}^n)$. *Then G has a fixed point, that is there is a point $p \in G$ such that $S(p) = p$ for all $S \in G$.*

Proof: Let $G = \{S_1, S_2, \cdots, S_m\} \leq \text{Isom}(\mathbb{R}^n)$. Fix a point $a \in \mathbb{R}^n$ and let $p = \frac{1}{m}\sum_{i=1}^{m} S_i(a)$. Let $k \in \{1, 2, \cdots, m\}$, and say $S_k = Ax = B$ with $A \in O_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. Since left multiplication by S_k is a permutation of G we have $G = \big\{S_kS_1, S_kS_2, \cdots, S_kS_m\big\}$, and so

$$\begin{aligned} S_k(p) &= S_k\Big(\tfrac{1}{m}\sum_{i=1}^{m} S_i(a)\Big) = A\Big(\tfrac{1}{m}\sum_{i=1}^{m} S_i(a)\Big) + b = \Big(\tfrac{1}{m}\sum_{i=1}^{m} AS_i(a)\Big) + b \\ &= \tfrac{1}{m}\sum_{i=1}^{m}\big(AS_i(a) + b\big) = \tfrac{1}{m}\sum_{i=1}^{m} S_kS_i(a) = \tfrac{1}{m}\sum_{j=1}^{m} S_j = p. \end{aligned}$$

Thus $S_k(p) = p$ for all indices k, as required.

7.12 Example: The following maps are all isometries on $\mathbb{R}^2$.

(1) The **identity** map is the map $I : \mathbb{R}^2 \to \mathbb{R}^2$ given by $I(x) = x$.
(2) For $u \in \mathbb{R}^2$, the **translation** by u is the map $T_u : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T_u(x) = x + u$.
(3) For $p \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$, the **rotation** about p by θ is the map $R_{p,\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$R_{p,\theta}(x) = p + R_\theta(x - p) \ \text{ where } \ R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(4) For a line L in $\mathbb{R}^2$, the **reflection** in L is the map $F_L : \mathbb{R}^2 \to \mathbb{R}^2$ which is given by any of the following three equivalent formulas. When L is the line in $\mathbb{R}^2$ through p perpendicular to u, we have

$$F_L(x) = x - \frac{2(x-p)\cdot u}{\|u\|^2}\,u.$$

When L is the line $ax + by + c = 0$, the above formula becomes

$$F_L(x, y) = (x, y) - \frac{2(ax + by + c)}{a^2 + b^2}\,(a, b).$$

When L is the line through p in the direction of the vector $\big(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\big)$, F_L is given by

$$F_L(x) = p + F_\theta(x - p) \ \text{ where } \ F_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

(5) For a vector $u \in \mathbb{R}^2$ and a line L in $\mathbb{R}^2$ which is parallel to u, the **glide reflection** $G_{u,L} : \mathbb{R}^2 \to \mathbb{R}^2$ is the composite

$$G_{u,L} = T_uF_L = F_LT_u$$

(when L is not parallel to u, the composites T_uF_L and F_LT_u are not equal, and they are not called glide reflections).

Of the above examples, the maps I, T_u and $R_{p,\theta}$ all preserve orientation, while the maps F_L and $G_{u,L}$ reverse orientation.

7.13 Theorem: (*Composites of Reflections in* $\mathbb{R}^2$) *Let* L *and* M *be lines in* $\mathbb{R}^2$.
(1) If $L=M$ *then* $F_MF_L=I$.
(2) If L *is parallel to* M *then* $F_MF_L=T_{2u}$ *where* u *is the vector from* L *orthogonally to* M.
(3) If $L\cap M=\{p\}$ *then* $F_MF_L=R_{p,2\theta}$ *where* θ *is the angle from* L *counterclockwise to* M.

Proof: Suppose first that $L=M$. Say L is the line through p in the direction of the vector $\left(\cos\frac{\theta}{2},\sin\frac{\theta}{2}\right)$ so that $F_L(x)=p+F_\theta(x-p)$. Then for all $x\in\mathbb{R}^2$ we have

$$F_LF_L(x)=F_L\big(p+F_\theta(x-p)\big)=p+F_\theta\big(F_\theta(x-p)\big)=p+x-p=x=I(x).$$

Next, suppose that L is parallel to M. Let u be the vector from L orthogonally to M, let $p\in L$ and let $q=p+u\in M$. Then for $x,y\in\mathbb{R}^2$ we have $F_L(x)=x-\frac{2(x-p)\,\bullet\,u}{\|u\|^2}\,u$ and $F_M(y)=y-\frac{2(y-p-u)\,\bullet\,u}{\|u\|^2}\,u$ and so

$$\begin{aligned}F_MF_L(x)&=F_M\left(x-\frac{2(x-p)\,\bullet\,u}{\|u\|^2}\,u\right)\\&=\left(x-\frac{2(x-p)\,\bullet\,u}{\|u\|^2}\,u\right)-\frac{2\left(x-p-u-\frac{2(x-p)\,\bullet\,u}{\|u\|^2}\,u\right)\bullet\,u}{\|u\|^2}\,u\\&=x-\frac{2(x-p)\,\bullet\,u}{\|u\|^2}\,u-\frac{2(x-p)\,\bullet\,u}{\|u\|^2}\,u+\frac{2u\,\bullet\,u}{\|u\|^2}\,u+\frac{4\big((x-p)\,\bullet\,u\big)(u\,\bullet\,u)}{\|u\|^4}\,u\\&=x+2u=T_{2u}(x).\end{aligned}$$

Finally, suppose that $L\cap M=\{p\}$. Say L is in the direction of $\left(\cos\frac{\alpha}{2},\sin\frac{\alpha}{2}\right)$ and M is in the direction of $\left(\cos\frac{\beta}{2},\sin\frac{\beta}{2}\right)$. Then for $x,y\in\mathbb{R}^2$ we have $F_L(x)=p+F_\alpha(x-p)$ and $F_M(y)=p+F_\beta(y-p)$ and so

$$F_MF_L(x)=F_M\big(p+F_\alpha(x-p)\big)=p+F_\beta\big(F_\alpha(x-p)\big)=p+R_{\beta-\alpha}(x-p)=R_{p,2\theta}(x)$$

where $\theta=\frac{\beta}{2}-\frac{\alpha}{2}$, which is the angle from L to M.

7.14 Theorem: (*The Geometric Classification of Isometries on* $\mathbb{R}^2$) *Every isometry on* $\mathbb{R}^2$ *is equal to one of the maps* I , T_u , $R_{p,\theta}$, F_L , $G_{u,L}$.

Proof: Let $S\in\text{Isom}(\mathbb{R}^2)$, say $S(x)=Ax+b$ with $A\in O_2(\mathbb{R})$ and $b\in\mathbb{R}^2$. Recall that the elements in $O_2(\mathbb{R})$ are the rotation and reflection matrices R_θ and F_θ, and so with $S=T_uR_\theta$ or $S=T_uF_\theta$ where $u=-b$. First suppose that $S=T_uR_\theta$. Let M be the line through the origin perpendicular to u. Let $L=R_{-\theta/2}(M)$ so that $F_MF_L=R_\theta$. Let $N=T_{u/2}(M)$ so that $T_u=F_NF_M$. Then $S=T_uR_\theta=F_NF_MF_MF_L=F_MF_L$. By the above theorem, S is equal to the identity, a translation, or a rotation.

Now suppose that $S=T_uF_\theta$. Let L be the line through the origin in the direction of $\left(\cos\frac{\theta}{2},\sin\frac{\theta}{2}\right)$ so that $F_\theta=F_L$. Let M be the line through the origin which is perpendicular to u and let $N=T_{u/2}(M)$ so that $T_u=F_nF_M$. Then we have $S=F_NF_MF_L$. Note that $F_MF_L=R_{2\alpha}$ where α is the angle from L to M. Let $N'=N$, let M' be the line through $(0,0)$ which is perpendicular to N', and let $L'=R_{-\alpha}(M')$ so that $F_{M'}F_{L'}=R_{2\alpha}$. Then $S=F_NF_MF_L=F_NR_{2\alpha}=F_{N'}F_{M'}F_{L'}=R_{p,\pi}F_{L'}$ where p is the point of intersection of M' and N' (which are perpendicular). Let $L''=L'$, let M'' be the line through p parallel to L' and let $N''=R_{p.\pi/2}(M'')$ so that $R_{p,\pi}=F_{N''}F_{M''}$. Then we have $S=R_{p,\pi}F_{L'}=F_{N''}F_{M''}F_{L''}$. Since L'' is parallel to M'' we have $F_{M''}F_{L''}=T_{2v}$ where v is the vector from L'' to M''. Since L'' and M'' are perpendicular to N'', the vector v is parallel to N'' and so $S=F_{N''}T_{2v}$ is a glide reflection (or a reflection when $v=0$).

7.15 Theorem: (*The Classification of Finite Groups of Isometries on* $\mathbb{R}^2$) *Every finite subgroup of* $\text{Isom}(\mathbb{R}^2)$ *is isomorphic to* C_n *or* D_n *for some* $n \in \mathbb{Z}^+$.

Proof: Let G be a finite subgroup of $\text{Isom}(\mathbb{R}^2)$. Note that G cannot contain a translation T_u with $u \neq 0$ or a glide reflection $G_{u,L}$ with $u \neq 0$ because T_u and $G_{u,L}$ both have infinite order. Similarly, G cannot contain a rotation $R_{p,\alpha}$ where α is an irrational multiple of 2π because such a rotation has infinite order. Note that if G contains two rotations $R_{p,\alpha}$ and $R_{q,\beta}$ then we must have $p = q$ because G has a fixed point and $R_{p,\alpha}$ only fixes the point p and $R_{q,\beta}$ only fixes the point q. Note that if G contains only one unique reflection, then G cannot contain a rotation because if we had $R_{p,\alpha} \in G$ and $F_L \in G$ then p would be the unique fixed point of G so we would have $p \in L$ and would follow that $R_{p,\alpha} F_L = F_M F_L F_L = F_M$ so that $F_M \in G$ where $M = R_{p,\alpha/2}(L)$ so that $F_M F_M = R_{p,\alpha}$. Note that if G contains two distinct reflections F_L and F_M then we cannot have L and M parallel (since if they were parallel then $F_M F_M$ would be a translation) so G contains the rotation $R_{p,\alpha} = F_M F_L$ where $L \cap M = \{p\}$, hence p is the unique fixed point of G, hence $p \in M$ for every line M such that $F_M \in G$.

We claim that if G contains a rotation, then the set R of all rotations in G is a cyclic subgroup of G. We have already seen that all rotations are about the same point, say p. Let α be the smallest positive real number for which $R_{p,\alpha} \in G$. Let $\beta = R_{p,\beta} \in G$. Write $\alpha = \frac{2\pi k}{n}$ and $\beta = \frac{2\pi l}{n}$. Write $l = qk + r$ with $0 \leq r < k$. Then for $\gamma = \frac{2\pi r}{n}$ we have $R_{p,\gamma} = R_{p,\beta}\big(R_{p,\alpha}\big)^{-q} \in G$ which implies that $r = 0$ by the minimality of α. Thus $R = \big\langle R_{p,\alpha} \big\rangle$.

Finally, not that if G contains a reflection F_L and a rotation, and the above group R is generated by $R_{p,\alpha}$, then G contains all the reflections $F_M = R_{p,k\alpha} F_L$ with $k \in \mathbb{Z}^+$ and no other reflections. Indeed, if $F_M \in G$ then we have already seen that $p \in M$ hence $F_M F_L$ is a rotation so we have $F_M F_L \in R = \big\langle R_{p,\alpha} \big\rangle$ so we have $F_M F_L = R_{p,k\alpha}$ for some $k \in \mathbb{Z}^+$ hence $F_M = R_{p,k\alpha} F_L$.

We summarize. If G contains no rotations and no reflections then $G = \{I\}$. If G contains only reflections then $G = \{I, F\}$ for some reflection F. If G contains only rotations then $G = \langle R \rangle$ for some rotation $R = R_{p,2\pi/n}$ with $n \in \mathbb{Z}^+$. If G contains a reflection and a rotation, then we have $G = \langle R, F \rangle = \big\{ R^k, R^k F \big| k \in \mathbb{Z}_n \big\}$ for some rotation of the form $R = R_{p,2\pi/n}$ and a reflection $F = F_L$ for some line L with $p \in L$. In the final case one can verify that $G \cong D_n$.

7.16 Example: The following maps are all isometries on $\mathbb{R}^3$.
(1) the **identity** map is the map $I:\mathbb{R}^3\to\mathbb{R}^3$ given by $I(x)=x$.
(2) For $u\in\mathbb{R}^3$, the **translation** by u is the map $T_u:\mathbb{R}^3\to\mathbb{R}^3$ given by $T_u(x)=x+u$.
(3) For a point $p\in\mathbb{R}^3$, a nonzero vector $0\neq u\in\mathbb{R}^3$ and an angle $\theta\in\mathbb{R}$ the **rotation** $R_{p,u,\theta}:\mathbb{R}^3\to\mathbb{R}^3$ is given by

$$R_{p,u,\theta}(x)=p+R_{u,\theta}(x-p)$$

where $R_{u,\theta}$ is the rotation in $\mathbb{R}^3$ about the vector u by the angle θ; if $\{u,v,w\}$ is a positively oriented orthogonal basis for $\mathbb{R}^3$ with all three vectors u, v and w of the same length, then $R=R_{u,\theta}$ is given by $R(u)=u$, $R(v)=(\cos\theta)v+(\sin\theta)w$ and $R(w)=-(\sin\theta)v+(\cos\theta)w$.
(4) For a point $p\in\mathbb{R}^3$, a nonzero vector $0\neq u\in\mathbb{R}^3$ and an angle $\theta\in\mathbb{R}$ the **twist** $W_{p,u,\theta}:\mathbb{R}^3\to\mathbb{R}^3$ is the composite $W_{p,u,\theta}=T_uR_{p,u,\theta}=R_{p,u,\theta}T_u$.
(5) For a plane P in $\mathbb{R}^3$, the **reflection** in P is the map $F_P:\mathbb{R}^3\to\mathbb{R}^3$ described in example 7.6.
(6) For a vector $u\in\mathbb{R}^3$ and a plane P in $\mathbb{R}^3$ which is parallel to u, the **glide reflection** $G_{u,P}:\mathbb{R}^3\to\mathbb{R}^3$ is the composite $G_{u,P}=T_uF_P=F_PT_u$.
(7) For a point $p\in\mathbb{R}^3$, a nonzero vector $0\neq u\in\mathbb{R}^3$ and an angle $\theta\in\mathbb{R}$, the **rotary reflection** $H_{p,u,\theta}:\mathbb{R}^3\to\mathbb{R}^3$ is the composite $H_{p,u,\theta}=R_{p,u,\theta}F_P=F_PR_{p,u,\theta}$ where P is the plane through p perpendicular to u.

7.17 Theorem: (*The Geometric Classification of Isometries in* $\mathbb{R}^3$) *Every isometry on* $\mathbb{R}^3$ *is equal one of the following*

$$I\,,\ T_u\,,\ R_{p,u,\theta}\,,\ W_{p,u,\theta}\,,\ F_P\,,\ G_{u,P}\,,\ H_{p,u,\theta}\,.$$

Proof: We omit the proof.

7.18 Theorem: (*The Classification of Finite Rotation Groups in* $\mathbb{R}^3$) *Every finite rotation group is isomorphic to one of the groups*

$$C_n\,,\ D_n\,,\ A_4\,,\ S_4\,,\ A_5\,.$$

Proof: We omit the proof.

7.19 Definition: Let X be a set and let $G\leq\mathrm{Perm}(X)$. For $f\in G$, the **fixed point set** of f is the set

$$\mathrm{Fix}(f)=\left\{x\in X\,\middle|\,f(x)=x\right\}\subseteq X.$$

For $a\in X$, the **orbit** of a under G is the set

$$\mathrm{Orb}(a)=\left\{f(a)\,\middle|\,f\in G\right\}\subseteq X.$$

Note that the distinct orbits are disjoint since for $a,b\in X$, if $b\in\mathrm{Orb}(a)$ with say $b=f(a)$ then we have $a\in\mathrm{Orb}(b)$ since $a=f^{-1}(b)$. The set of distinct orbits is denoted by X/G so we have

$$X/G=\left\{\mathrm{Orb}(a)\,\middle|\,a\in X\right\}.$$

For $a\in X$, the **stabilizer** of a in G is the subgroup

$$\mathrm{Stab}(a)=\left\{f\in G\,\middle|\,f(a)=a\right\}\leq G.$$

Note that $\mathrm{Stab}(a)$ is a subgroup of G because $I(a)=a$ so that $I\in\mathrm{Stab}(a)$, if $f,g\in\mathrm{Stab}(a)$ then $(fg)(a)=f(g(a))=f(a)=a$ so that $fg\in\mathrm{Stab}(a)$, and if $f\in\mathrm{Stab}(a)$ then $f^{-1}(a)=f^{-1}(f(a))=a$ so that $f^{-1}\in\mathrm{Stab}(a)$.

7.20 Theorem: (*The Orbit/Stabilizer Theorem*) *Let X be a set and let G be a finite subgroup of* $\mathrm{Perm}(X)$. *Then for all $a \in X$ we have*

$$|G| = \big|\mathrm{Orb}(a)\big|\big|\mathrm{Stab}(a)\big|.$$

Proof: Let $a \in X$. Let $H = \mathrm{Stab}(a) \leq G$. Define $\Phi : G/H \to \mathrm{Orb}(a)$ by $\Phi(fH) = f(a)$. Note that Φ is well defined because for $f, g \in G$ we have

$$fH = gH \implies g^{-1}f \in H \implies g^{-1}f(a) = a \implies f(a) = g(a) \implies \Phi(fH) = \Phi(gH)\,.$$

Note that Φ is injective because for $f, g \in G$ we have

$$\Phi(fH) = \Phi(gH) \implies f(a) = g(a) \implies g^{-1}f(a) = a \implies g^{-1}f \in H \implies fH = gH\,.$$

Finally, note that Φ is clearly surjective.

7.21 Theorem: (*The Burnside-Cauchy-Frobenius Lemma*) *Let X be a set and let G be a finite subgroup of* $\mathrm{Perm}(X)$. *Then*

$$|G|\,\big|X/G\big| = \sum_{a\in G} \big|\mathrm{Fix}(a)\big|\,.$$

Proof: Let $T = \big\{(f,a)\big|f \in G, a \in X, f(a) = a\big\}$. Then we have

$$|T| = \sum_{f\in G} \big|\{a \in X|f(a) = a\}\big| = \sum_{f\in G} \big|\mathrm{Fix}(f)\big|$$

and we have

$$\begin{aligned}|T| &= \sum_{a\in X} \big|\{f \in G\big|f(a) = a\}\big| = \sum_{a\in X} \big|\mathrm{Stab}(a)\big| = \sum_{a\in X} \frac{|G|}{|\mathrm{Orb}(a)|} \\ &= |G| \sum_{a\in X} \frac{1}{|\mathrm{Orb}(a)|} = |G| \sum_{A\in X/G} \sum_{a\in A} \frac{1}{|A|} = |G| \sum_{A\in X/G} 1 = |G|\big|X/G\big|\,.\end{aligned}$$

7.22 Example: In how many ways (up to symmetry under the symmetry group D_6) can we colour the vertices of the regular hexagon C_6 using 3 colours?

Solution: Let X be the set of possible colourings without considering symmetry under D_6, and note that $|X| = 3^6$. Each element of D_6 permutes the vertices of C_6 and hence permutes the elements of X, and in this way we identify D_6 with a subgroup of $\mathrm{Perm}(X)$. We make a table showing $\big|\mathrm{Fix}(A)\big|$ for each $A \in D_6 \leq \mathrm{Perm}(X)$.

A	# of such A	$\big\lvert\mathrm{Fix}(A)\big\rvert$
I	1	3^6
R_3	1	3^3
R_2, R_4	2	3^2
R_1, R_5	2	3^1
F_0, F_2, F_4	3	3^4
F_1, F_3, F_5	3	3^3

The number of colourings up to D_6 symmetry is equal to the number of orbits, which is

$$\big|X/D_6\big| = \frac{1}{|D_6|} \sum_{A\in D_6} \big|\mathrm{Fix}(A)\big| = \tfrac{1}{12}\big(3^6 + 3^3 + 2\cdot 3^2 + 2\cdot 3^1 + 3\cdot 3^4 + 3\cdot 3^2\big) = 92\,.$$

7.23 Example: Let G be the rotation group of a cube Q. In how many ways (up to symmetry under G) can we colour the 8 vertices of Q using 2 colours?