PMATH 336 Introduction to Group Theory, Solutions to the Exercises for Chapter 2

1: In D_n, for $k \in \mathbb{Z}_n$, we write R_k for the rotation in about the point $(0,0)$ by the angle $\frac{2\pi k}{n}$, and we write F_k for the reflection in the line through $(0,0)$ and $\left(\cos\frac{\pi k}{n}, \sin\frac{\pi k}{n}\right)$.

(a) Find all values of $k \in \mathbb{Z}_6$ such that $F_3 R_k F_1 = R_k$ in D_6.

Solution: We can use the formulas $R_k R_l = R_{k+l}$, $F_k F_l = R_{k-l}$, $R_k F_l = F_{k+l}$ and $F_k R_l = F_{k-l}$. For $k \in \mathbb{Z}_6$, $F_3 R_k F_1 = R_k \iff F_{3-k} F_1 = R_k \iff R_{3-k-1} = R_k \iff 3-k-1 = k \iff 2k = 2 \iff k = 1$ or 4.

(b) Find the centralizer of F_1 in D_6.

Solution: $R_k F_1 = F_1 R_k \iff F_{k+1} = F_{1-k} \iff k+1 = 1-k \iff 2k = 0 \iff k = 0$ or 3 in $\mathbb{Z}_6$. Also, $F_k F_1 = F_1 F_k \iff R_{k-1} = R_{1-k} \iff k-1 = 1-k \iff 2k = 2 \iff k = 1$ or 4 in $\mathbb{Z}_6$. So $C(F_1) = \{I, R_3, F_1, F_4\}$.

2: (a) Find $\left|GL(3,\mathbb{Z}_2)\right|$

Solution: For a matrix in $GL(3,\mathbb{Z}_2)$, the first row must be non-zero, and there are $2^3 - 1 = 7$ such rows. Having fixed the first row, the second row can be any row that is not a multiple of the first; there are 2^1 multiples of the first row, so there are $2^3 - 2^1 = 6$ possibilities for the second row. Having fixed the first two rows, the last row can be any row which is not a linear combination of the first two rows; there are 2^2 different linear combinations of the first two rows, so there are $2^3 - 2^2 = 4$ possibilities for the last row. Altogether, there are $7 \cdot 6 \cdot 4 = 168$ matrices in $GL(3,\mathbb{Z}_2)$, so $\left|GL(3,\mathbb{Z}_2)\right| = 168$.

(b) List all the elements in $SO(3,\mathbb{Z}_2)$.

Solution: Let A be the 3×3 matrix over $\mathbb{Z}_2$ with columns u_1, u_2, u_3. Note that $A \in SO(3,\mathbb{Z}_2) = O(3,\mathbb{Z}_2) \iff A^T A = I \iff \left(u_k \bullet u_k = 1 \text{ for all } k \text{ and } u_k \bullet u_l = 0 \text{ for all } k \neq l\right)$. The only vectors u_k with $u_k \bullet u_k = 1$ are the three standard basis vectors e_k and the vector $(1,1,1)^T$, so each u_k must be one of these 4 vectors. Note that if $u_k = (1,1,1)^T$ and u_l is any one of the above 4 vectors then $u_k \bullet u_l = 1$, not 0, so we cannot have $u_k = (1,1,1)^T$. Also, the vectors u_k must be distinct, so the 3 vectors u_k are equal to the 3 standard basis vectors (in some order). Thus there are 6 matrices in $SO(3,\mathbb{Z}_2)$, namely

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \text{ and } \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.$$

3: (a) Show that U_{26} is cyclic.

Solution: Notice that $\langle 7\rangle = \{1,7,23,5,9,11,25,19,3,21,17,15\} = U_{26}$, and so U_{26} is cyclic.

(b) List all the elements and all the generators in every subgroup of U_{26}.

Solution: We list all the subgroups with the generators in boldface.

$\langle 7\rangle = \{1,\mathbf{7},23,5,9,\mathbf{11},25,\mathbf{19},3,21,17,\mathbf{15}\}$

$\langle 7^2\rangle = \{1,\mathbf{23},9,25,3,\mathbf{17}\}$

$\langle 7^3\rangle = \{1,\mathbf{5},25,\mathbf{21}\}$

$\langle 7^4\rangle = \{1,\mathbf{9},\mathbf{3}\}$

$\langle 7^6\rangle = \{1,\mathbf{25}\}$

$\langle 7^{12}\rangle = \{\mathbf{1}\}$

4: (a) Determine the number of subgroups of $\mathbb{Z}_{12,000}$.

Solution: Since $12,000 = 2^5 3^1 5^3$, the divisors of $12,000$ are of the form $2^i 3^j 5^k$ with $0 \leq i \leq 5$, $0 \leq j \leq 1$ and $0 \leq k \leq 3$. Since there are 6 possible values for i, 2 possible values for j and 4 posible values for k, there are $6 \cdot 2 \cdot 4 = 48$ divisors of $12,000$. Thus there are 48 subgroups of $\mathbb{Z}_{12,000}$.

(b) Find the number of elements of even order in $\mathbb{Z}_{12,000}$.

Solution: The odd factors of 12,000 are of the form $3^j 5^k$ with $0 \leq j \leq 1$ and $0 \leq k \leq 3$. There are 8 such odd factors, namely 1, 5, 25, 125, 3, 15, 75 and 375, and correspondingly there are 8 subgroups of odd order in $\mathbb{Z}_{12,000}$. The elements in $\mathbb{Z}_{12,000}$ of odd order are the generators of these 8 subgroups, so the number of elements of odd order is $\phi(1) + \phi(5) + \phi(25) + \phi(125) + \phi(3) + \phi(15) + \phi(75) + \phi(375) = 1 + 4 + 20 + 100 + 2 + 8 + 40 + 200 = 375$. Thus the number of elements of even order in $\mathbb{Z}_{12,000}$ is $12,000 - 375 = 825$.

5: (a) Find the number of elements of each order in $\mathbb{Z}_3 \times \mathbb{Z}_6$.

Solution: There is 1 element of order 1, 1 of order 2, 8 of order 3 and 8 of order 6.

(b) List all the elements in every cyclic subgroup of $\mathbb{Z}_3 \times \mathbb{Z}_6$.

Solution: By the result of Part (a), there is 1 cyclic subgroup of order 1, 1 of order 2, 4 of order 3 and 4 of order 6. The cyclic subgroups are

$$
\begin{aligned}
\langle(0,0)\rangle &= \big\{(0,0)\big\} \\
\langle(0,3)\rangle &= \big\{(0,0),(0,3)\big\} \\
\langle(0,2)\rangle &= \big\{(0,0),(0,2),(0,4)\big\} \\
\langle(1,0)\rangle &= \big\{(0,0),(1,0),(2,0)\big\} \\
\langle(1,2)\rangle &= \big\{(0,0),(1,2),(2,4)\big\} \\
\langle(1,4)\rangle &= \big\{(0,0),(1,4),(2,2)\big\} \\
\langle(0,1)\rangle &= \big\{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5)\big\} \\
\langle(1,3)\rangle &= \big\{(0,0),(1,3),(2,0),(0,3),(1,0),(2,3)\big\} \\
\langle(1,1)\rangle &= \big\{(0,0),(1,1),(2,2),(0,3),(1,4),(2,5)\big\} \\
\langle(1,5)\rangle &= \big\{(0,0),(1,5),(2,4),(0,3),(1,2),(5,1)\big\}
\end{aligned}
$$

(c) List all the elements in every non-cyclic subgroup of $\mathbb{Z}_3 \times \mathbb{Z}_6$. Explain why your list is complete.

Solution: $\mathbb{Z}_3 \times \mathbb{Z}_6$ and the subgroup $\mathbb{Z}_3 \times \langle 2 \rangle = \big\{(0,0),(1,0),(2,0),(0,2),(1,2),(2,2),(0,4),(1,4),(2,4)\big\}$ are non-cyclic subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_6$. We now show that these are the only two. Since $\big|\mathbb{Z}_3 \times \mathbb{Z}_6\big| = 18$, any subgroup must have order 1, 2, 3, 6, 9 or 18. Any group of order 1, 2 or 3 must be cyclic, and any abelian group of order 6 is cyclic, so any non-cyclic subgroup of $\mathbb{Z}_3 \times \mathbb{Z}_6$ must have order 9 or 18. Of course, the only subgroup of $\mathbb{Z}_3 \times \mathbb{Z}_6$ of order 18 is $\mathbb{Z}_3 \times \mathbb{Z}_6$ itself. Now, let H be a subgroup of order 9 in $\mathbb{Z}_3 \times \mathbb{Z}_6$. Then the elements of H could only be of order 1, 3 or 9. No elements in $\mathbb{Z}_3 \times \mathbb{Z}_6$ have order 9, so H must consist of the identity along with 8 elements of order 3. But the group $\mathbb{Z}_3 \times \mathbb{Z}_6$ only has 8 elements of order 3, so all of these must be in H, and hence H must be the group $\mathbb{Z}_3 \times \langle 2 \rangle$.