PMATH 336 Introduction to Group Theory, Solutions to the Exercises for Chapter 3

1: In S_8, let $\alpha = (1632)(27)(3748)$ and let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 7 & 2 & 8 & 4 & 1 & 6 \end{pmatrix}$.

(a) Find $|\alpha|$ and find $(-1)^\beta$.

Solution: First we express α and β as products of disjoint cycles. We find that $\alpha = (163)(2748)$ and $\beta = (137)(25864)$. So $|\alpha| = \text{lcm}\,(3,4) = 12$ and $(-1)^\beta = (-1)^{4+6} = 1$.

(b) Express each of the permutations α^{110} and $\alpha\beta\alpha^{-1}$ as products of disjoint cycles.

Solution: We have $\alpha^{110} = \alpha^{9\cdot 12+8} = (\alpha^{12})^9\alpha^2 = \alpha^2 = (163)^2(2748)^2 = (136)(24)(78)$, and we have $\alpha\beta\alpha^{-1} = (163)(2748)(137)(25864)(136)(2847) = (146)(23875)$.

2: (a) Find the number of elements of each order in S_7 and in A_7.

Solution: We find the number of permutations of each form, them we list the number of each order.

form of α	$\lvert\alpha\rvert$	$(-1)^\alpha$	# of such α
(a)	1	$+$	1
(ab)	2	$-$	$\binom{7}{2} = 21$
$(ab)(cd)$	2	$+$	$\binom{7}{4}\cdot 3 = 105$
$(ab)(cd)(ef)$	2	$-$	$\binom{7}{6}\cdot 5\cdot 3 = 105$
(abc)	3	$+$	$\binom{7}{3}\cdot 2 = 70$
$(abc)(de)$	6	$-$	$\binom{7}{3}\cdot 2\cdot\binom{4}{2} = 420$
$(abc)(de)(fg)$	6	$+$	$\binom{7}{3}\cdot 2\cdot\cdot 3 = 210$
$(abc)(def)$	3	$+$	$\binom{7}{6}\cdot 5\cdot 4\cdot 2 = 280$
$(abcd)$	4	$-$	$\binom{7}{4}\cdot 3\cdot 2 = 210$
$(abcd)(ef)$	4	$+$	$\binom{7}{4}\cdot 3\cdot 2\cdot\binom{3}{2} = 630$
$(abcd)(efg)$	12	$-$	$\binom{7}{4}\cdot 3\cdot 2\cdot 2 = 420$
$(abcde)$	5	$+$	$\binom{7}{5}\cdot 4! = 504$
$(abcde)(fg)$	10	$-$	$\binom{7}{5}\cdot 4! = 504$
$(abcdef)$	6	$-$	$\binom{7}{6}\cdot 5! = 840$
$(abcdefg)$	7	$+$	$6! = 720$

In S_7:

order	#
1	1
2	231
3	350
4	840
5	504
6	1470
7	720
10	504
12	420

In A_7:

order	#
1	1
2	105
3	350
4	630
5	504
6	210
7	720

(b) Find the number of cyclic subgroups of A_7.

Solution: Recall that the number of cyclic subgroups of order k is equal to the number of elements of order k divided by $\phi(k)$. So from the third of the tables in part (a), we see that the total number of cyclic subgroups is $\frac{1}{\phi(1)} + \frac{105}{\phi(2)} + \frac{350}{\phi(3)} + \frac{630}{\phi(4)} + \frac{504}{\phi(5)} + \frac{210}{\phi(6)} + \frac{720}{\phi(7)} = 1 + 105 + 175 + 315 + 126 + 105 + 120 = 947$.

3: Let $n \geq 3$.

(a) Show that $Z(S_n) = \{e\}$.

Solution: Suppose that $\alpha \neq e$, and say the permutation α sends k to l, where $k \neq l$. Choose $m \notin \{k, l\}$. Then $(lm)\alpha$ sends k to m, but $\alpha(lm)$ sends k to l, so $(lm)\alpha \neq \alpha(lm)$, and therefore $\alpha \notin Z(S_n)$.

(b) Show that every element in A_n is equal to a product of 3-cycles.

Solution: We already know that every permutation in A_n is equal to a product of an even number of 2-cycles, so it suffices to show that every product of a pair of 2-cycles is equal to a product of 3-cycles. Every product of a pair of 2-cycles is of one of the following three forms, where a, b, c and d are distinct: $(ab)(ab)$, $(ab)(ac)$ or $(ab)(cd)$, and indeed, each of these can be written as a product of 3-cycles:

$$\begin{aligned}(ab)(ab) &= (abc)(acb)\\ (ab)(ac) &= (acb)\\ (ab)(cd) &= (adc)(abc)\,.\end{aligned}$$