PMATH 336 Introduction to Group Theory, Solutions to the Exercises for Chapter 4

1: (a) Define $\phi : \mathbb{Z}_{60} \to U_{45}$ by $\phi(k) = 2^k$. Show that ϕ is a group homomorphism, and find $\mathrm{Ker}(\phi)$ and $\mathrm{Im}(\phi)$.

Solution: In U_{45} we have $\langle 2 \rangle = \{1, 2, 4, 8, 16, 32, 19, 38, 31, 17, 34, 23\}$. Since $|2| = 12$ and 12 is a factor of 60, ϕ is well defined (that is, $k = l \bmod 60 \Longrightarrow 2^k = 2^l \bmod 45$). ϕ is a homomorphism since $\phi(k + l) = 2^{k+l} = 2^k 2^l = \phi(k)\phi(l)$. The image of ϕ is $\mathrm{Im}(\phi) = \langle 2 \rangle$. The kernel of ϕ is $\ker(\phi) = \langle 12 \rangle = \{0, 12, 24, 36, 48\}$.

(b) Define $\psi : SL(n, \mathbb{R}) \times \mathbb{R}^* \to GL(n, \mathbb{R})$ by $\psi(A, t) = tA$. Show that ψ is a group homomorphism and find $\mathrm{Ker}(\psi)$ and $\mathrm{Im}(\psi)$.

Solution: $\psi\big((A, s) \cdot (B, t)\big) = \psi(AB, st) = st\,AB = (s\,A)(t\,B) = \psi(A)\phi(B)$, so ψ is a homomorphism. $\mathrm{Ker}(\psi) = \{(A, t) | tA = I\}$. If $tA = I$ then $t^n \det A = \det I = 1$, so when $\det A = 1$ we have $t^n = 1$: when n is odd we have $t = 1$ and $A = I$, and when n is even we have $t = \pm 1$ and $A = \pm I$. Thus $\mathrm{Ker}(\psi) = \{(I, 1)\}$ when n is odd, and $\mathrm{Ker}(\psi) = \{(I, 1), (-I, -1)\}$ when n is even. The image of ψ is $\mathrm{Im}(\psi) = \{tA\}$. Again notice that $\det tA = t^n \det A = t^n$, so when n is even we have $\det tA > 0$. We can see that $\mathrm{Im}(\psi) = GL_+(n, \mathbb{R})$ (the group of $n \times n$ matrices with positive determinant) when n is even and $\mathrm{Im}(\psi) = GL(n, \mathbb{R})$ when n is odd, because given any matrix B in $GL_+(n, \mathbb{R})$ (when n is even) or in $GL(n, \mathbb{R})$ (when n is odd), we can let $t = \sqrt[n]{\det B}$ and then let $A = \frac{1}{t} B$ and then we will have $\psi(A, t) = B$.

2: Show that no two of the groups $\mathbb{Z}_8$, U_{16}, D_4 and ${\mathbb{Z}_2}^3$ are isomorphic.

Solution: We list the orders of all the elements in each of these groups (and also the quaternionic group Q).

In $\mathbb{Z}_8$:

x	0	1	2	3	4	5	6	7
$\lvert x\rvert$	1	8	4	8	2	8	4	8

In U_{16}:

x	1	3	5	7	9	11	13	15
$\lvert x\rvert$	1	4	4	2	2	4	4	2

In D_4:

x	I	R_1	R_2	R_3	F_0	F_1	F_2	F_3
$\lvert x\rvert$	1	4	2	4	2	2	2	2

In ${\mathbb{Z}_2}^3$:

x	$(0,0,0)$	$(0,0,1)$	$(0,1,0)$	$(0,1,1)$	$(1,0,0)$	$(1,0,1)$	$(1,1,0)$	$(1,1,1)$
$\lvert x\rvert$	1	2	2	2	2	2	2	2

In Q:

x	1	i	j	k	-1	$-i$	$-j$	$-k$
$\lvert x\rvert$	1	4	4	4	2	4	4	4

Since no two of these groups have the same number of elements of each order, no two of them are isomorphic.

3: Find the number of elements of each order in $U(55) \times A_4$.

Solution: We know that $U(55) \cong U(5) \times U(11) \cong \mathbb{Z}_4 \times \mathbb{Z}_{10}$, so we make a table to determine the number of elements of each order in $U(55)$, then a similar table for $U(55) \times A_4$, and then a third table to summarize.

$\mathbb{Z}_4$		$\mathbb{Z}_{10}$		$U(55)$	
$\lvert a\rvert$	#	$\lvert b\rvert$	#	$\lvert(a,b)\rvert$	#
1	1	1	1	1	1
		2	1	2	1
		5	4	5	4
		10	4	10	4
2	1	1	1	2	1
		2	1	2	1
		5	4	10	4
		10	4	10	4
4	2	1	1	4	2
		2	1	4	2
		5	4	20	8
		10	4	20	8

$U(55)$		A_4		$U(55) \times A_4$	
$\lvert a\rvert$	#	$\lvert b\rvert$	#	$\lvert(a,b)\rvert$	#
1	1	1	1	1	1
		2	3	2	3
		3	8	3	8
2	3	1	1	2	3
		2	3	2	9
		3	8	6	24
4	4	1	1	4	4
		2	3	4	12
		3	8	12	32
5	4	1	1	5	4
		2	3	10	12
		3	8	15	32
10	12	1	1	10	12
		2	3	10	36
		3	8	30	96
20	16	1	1	20	16
		2	3	20	48
		3	8	60	128

$U(55) \times A_4$	
$\lvert a\rvert$	#
1	1
2	15
3	8
4	16
5	4
6	24
10	60
12	32
15	32
20	64
30	96
60	128

4: (a) Find the number of homomorphisms from $\mathbb{Z}_{12}$ to D_9.

Solution: There are 12 homomorphisms from $\mathbb{Z}_{12}$ to D_9 because there are 12 elements a in D_9 with $|a|\big|12$; the homomorphisms are the maps $\phi_a(k) = a^k$ where $a = I, R_3, R_6$ or F_k, $k = 0, 1, \cdots 8$.

(b) Find the number of homomorphisms from D_9 to $\mathbb{Z}_{12}$.

Solution: Let $\phi : D_9 \to \mathbb{Z}_{12}$ be a homomorphism. Note that ϕ is completely determined by the values $\phi(R_1)$ and $\phi(F_0)$, since $\phi(R_k) = \phi({R_1}^k) = k\phi(R_1)$ and $\phi(F_k) = \phi({R_1}^k F_0) = k\phi(R_1) + \phi(F_0)$. Since $|F_0| = 2$, $|\phi(F_0)|$ must be a factor of 2 and so $\phi(F_0) = 0$ or 6. Also, $F_1 = R_1F_0 = F_0R_8 \Longrightarrow \phi(R_1F_0) = \phi(F_0{R_1}^8) \Longrightarrow \phi(R_1) + \phi(F_0) = \phi(F_0) + 8\phi(R_1) \Longrightarrow 7\phi(R_1) = 0 \Longrightarrow \phi(R_1) = 0$. Thus there are only two homomorphisms, namely the identity and the homomorphism ϕ given by $\phi(R_k) = 0$ and $\phi(F_k) = 6$ for all k.

5: (a) Find the number of homomorphisms from $\mathbb{Z}_4 \times \mathbb{Z}_6$ to itself.

Solution: First we count the number of elements of each order in $\mathbb{Z}_4 \times \mathbb{Z}_6$.

$\mathbb{Z}_4$		$\mathbb{Z}_6$		$\mathbb{Z}_4 \times \mathbb{Z}_6$	
$\lvert a\rvert$	#	$\lvert b\rvert$	#	$\lvert(a,b)\rvert$	#
1	1	1	1	1	1
		2	1	2	1
		3	2	3	2
		6	2	6	2
2	1	1	1	2	1
		2	1	2	1
		3	2	6	2
		6	2	6	2
4	2	1	1	4	2
		2	1	4	2
		3	2	12	4
		6	2	12	4

$\mathbb{Z}_4 \times \mathbb{Z}_6$	
order	#
1	1
2	3
3	2
4	4
6	6
12	8

Since $|(1,0)| = 4$, $|\phi(1,0)| = 0$, 1, 2 or 4, so there are $1+3+4 = 8$ possibilities for $\phi(1,0)$. Since $|(0,1)| = 6$, $|\phi(0,1)| = 1$, 2, 3 or 6, so there are 1+3+2+6=12 possibilities for $\phi(0,1)$. Thus there are $8 \cdot 12 = 96$ homomorphisms from $\mathbb{Z}_4 \times \mathbb{Z}_6$ to itself.

(b) Find the number of homomorphisms from $\mathbb{Z}_4 \times \mathbb{Z}_6$ to D_{12}.

Solution: The homomorphisms from $\mathbb{Z}_4 \times \mathbb{Z}_6$ to D_{12} are the maps ϕ_{ab} given by $\phi_{ab}(k,l) = ka + lb$ where $a, b \in D_{12}$ with $a^4 = I$, $b^6 = I$ and $ab = ba$. We have $a^4 = I$ when $a = I$, R_3, R_6, R_9 or F_k for some k (there are 16 possibilities). We have $b^6 = I$ when $b = I$, R_2, R_4, R_6, R_8, R_{10} or F_k for some k (there are 18 possibilities). When $a = I$ or R_6, all 18 possibilities for b give $ab = ba$. When $a = R_3$ or R_9, then we have $ab = ba$ when $b = I$, R_2, R_4, R_6, R_8 or R_{10}, so there are 6 possibilities for b. When $a = F_k$ we have $ab = ba$ when $b = I$, R_6, F_k or F_{k+6}, so there are 4 possibilities for b. Thus there are $2 \cdot 18 + 2 \cdot 6 + 12 \cdot 4 = 96$ homomorphisms from $\mathbb{Z}_4 \times \mathbb{Z}_6$ to D_{12}.

6: Let $f : S_3 \to \{1, 2, 3, 4, 5, 6\}$ be the bijection given by the table of values

α	(1)	(12)	(13)	(23)	(123)	(132)
$f(\alpha)$	1	2	3	4	5	6

and let $\phi : S_3 \to S_6$ be the isomorphism given by $\phi(\alpha) = f \circ L_\alpha \circ f^{-1}$, where $L_\alpha(\beta) = \alpha\beta$ for all $\beta \in S_3$. List all the elements in $\phi(S_3)$

Solution: Each row of the multiplication table of S_3 is a permutation of the elements of S_3, which corresponds, under f, to a permutation of $\{1, 2, 3, 4, 5, 6\}$. We list these permutations, and write them in cycle notation.

	(1)	(12)	(13)	(23)	(123)	(132)	1	2	3	4	5	6	
(1)	(1)	(12)	(13)	(23)	(123)	(132)	1	2	3	4	5	6	(1)
(12)	(12)	(1)	(132)	(123)	(23)	(13)	2	1	6	5	4	3	(12)(36)(45)
(13)	(13)	(123)	(1)	(132)	(12)	(23)	3	5	1	6	2	4	(13)(25)(46)
(23)	(23)	(132)	(123)	(1)	(13)	(12)	4	6	5	1	3	2	(14)(26)(35)
(123)	(123)	(13)	(23)	(12)	(132)	(1)	5	3	4	2	6	1	(156)(234)
(132)	(132)	(23)	(12)	(13)	(1)	(123)	6	4	2	3	1	5	(165)(243)

Thus $\phi(S_3) = \{(1), (12)(36)(45), (13)(25)(46), (14)(26)(35), (156)(234), (165)(243)\}$.

7: Find $\big|\mathrm{Inn}(Q)\big|$, where $Q = \big\{1, i, j, k, -1, -i, -j, -k\big\}$ is the **quaternionic** group, which has the following multiplication table $\big(Q$ is not isomorphic to any group from Exercise 2$\big)$.

	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	i	j	k	-1	$-i$	$-j$	$-k$
i	i	-1	k	$-j$	$-i$	1	$-k$	j
j	j	$-k$	-1	i	$-j$	k	1	$-i$
k	k	j	$-i$	-1	$-k$	$-j$	i	1
-1	-1	$-i$	$-j$	$-k$	1	i	j	k
$-i$	$-i$	1	$-k$	j	i	-1	k	$-j$
$-j$	$-j$	k	1	$-i$	j	$-k$	-1	i
$-k$	$-k$	$-j$	i	1	k	j	$-i$	-1

Solution: We make the conjugation table for Q, which lists the value of aba^{-1} for each pair $a, b \in Q$.

$a\backslash b$	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	i	j	k	-1	$-i$	$-j$	$-k$
i	1	i	$-j$	$-k$	-1	$-i$	j	k
j	1	$-i$	j	$-k$	-1	i	$-j$	k
k	1	$-i$	$-j$	k	-1	i	j	$-k$
-1	1	i	j	k	-1	$-i$	$-j$	$-k$
$-i$	1	i	$-j$	$-k$	-1	$-i$	j	k
$-j$	1	$-i$	j	$-k$	-1	i	$-j$	k
$-k$	1	$-i$	$-j$	k	-1	i	j	$-k$

The first four rows are distinct, so the inner automorphisms C_1, C_i, C_j and C_k are distinct, but each of the bottom four rows is the same as the row 4 rows above it, so $\mathrm{Inn}\,(Q) = \big\{I, C_i, C_j, C_k\big\}$ and $\big|\mathrm{Inn}\,(Q)\big| = 4$.