PMATH 336 Introduction to Group Theory, Solutions to the Exercises for Chapter 5

1: (a) List all the elements in every coset of $H = \{1, 9, 17, 33\}$ in $G = U_{40}$.

Solution: Since $|H| = 4$ and $|G| = 16$, we know there must $|G|/|H| = 4$ distinct cosets. They are $1H = \{1, 9, 17, 33\}$, $3H = \{3, 27, 11, 19\}$, $39H = (-1)H = \{-1, -9, -17, -33\} = \{39, 31, 23, 7\}$ and $37H = (-3)H = \{-3, -27, -11, -19\} = \{37, 13, 29, 21\}$.

(b) List all the elements in every left coset and every right coset of $H = \langle(1234)\rangle$ in $G = S_4$.

Solution: $G=\big\{(1), (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23), (123), (124), (132), (134), (142), (143), (234), (243), (1234), (1243), (1324), (1342), (1423), (1432)\big\}$ and $H = \big\{(1), (1234), (13)(24), (1432)\big\}$, and there are 24/4=6 left cosets;

$$\begin{aligned}
(1)H &= \big\{(1), (1234), (13)(24), (1432)\big\} \\
(12)H &= \big\{(12), (234), (1324), (143)\big\} \\
(13)H &= \big\{(13), (12)(34), (24), (14)(23)\big\} \\
(14)H &= \big\{(14), (123), (1342), (243)\big\} \\
(23)H &= \big\{(23), (134), (1243), (142)\big\} \\
(34)H &= \big\{(34), (124), (1423), (132)\big\}
\end{aligned}$$

There are also 6 right cosets:

$$\begin{aligned}
H(1) &= \big\{(1), (1234), (13)(24), (1432)\big\} \\
H(12) &= \big\{(12), (134), (1423), (243)\big\} \\
H(13) &= \big\{(13), (14)(23), (24), (12)(34)\big\} \\
H(14) &= \big\{(14), (234), (1243), (132)\big\} \\
H(23) &= \big\{(23), (124), (1342), (143)\big\} \\
H(34) &= \big\{(34), (123), (1324), (142)\big\}
\end{aligned}$$

2: Find four distinct subgroups $G_i \le S_5$, with $i = 1, 2, 3, 4$, such that $\mathrm{orb}_{G_i}(1) = \{1, 2, 4\}$. For each of the four subgroups G_i, find $\mathrm{stab}_{G_i}(1)$.

Solution: The groups G_i will permute the numbers 1, 2 and 4 amongst one another, and might also permute the numbers 3 and 5 amongst each other. The four groups are

$$\begin{aligned}
G_1 &= \big\{(1), (124), (142)\big\} \\
G_2 &= \big\{(1), (124), (142), (12), (14), (24)\big\} \\
G_3 &= \big\{(1), (124), (142), (35), (124)(35), (142)(35)\big\} \\
G_4 &= \big\{(1), (124), (142), (12), (14), (24), (35), (124)(35), (142)(35), (12)(35), (14)(35), (24)(35)\big\}
\end{aligned}$$

and the stabilizers are the groups $\mathrm{stab}_{G_1}(1) = \big\{(1)\big\}$, $\mathrm{stab}_{G_2}(1) = \big\{(1), (24)\big\}$, $\mathrm{stab}_{G_3}(1) = \big\{(1), (35)\big\}$ and $\mathrm{stab}(1) = \big\{(1), (24), (35), (24)(35)\big\}$.

3: Let $G=\mathbb{Z}^2$ and let $H=\big\langle(3,2),(6,8)\big\rangle=\big\{k(3,2)+l(6,8)\big|k,l\in\mathbb{Z}\big\}$. For each pair $(a,b)\in G$ with $0\leq a<6$ and $0\leq b<2$, find the order of $(a,b)+H$ in the group G/H.

Solution: The order of $(a,b)+H$ in G/H is the smallest positive integer n such that $n(a,b)\in H$, so we would like to find an easy way to test a given point (c,d) to determine whether it is in H. One way is to use a picture of the group H in G to determine whether a given point $(c,d)\in H$. Another is as follows.

$$\begin{aligned}
(c,d)\in H &\iff (c,d)=k(3,2)+l(6,8)\quad\text{for some } (k,l)\in\mathbb{Z}^2\\
&\iff \begin{pmatrix}3&6\\2&8\end{pmatrix}\begin{pmatrix}k\\l\end{pmatrix}=\begin{pmatrix}c\\d\end{pmatrix}\quad\text{for some } (k,l)\in\mathbb{Z}^2\\
&\iff \begin{pmatrix}k\\l\end{pmatrix}=\begin{pmatrix}3&6\\2&8\end{pmatrix}^{-1}\begin{pmatrix}c\\d\end{pmatrix}=\tfrac{1}{12}\begin{pmatrix}8&-6\\-2&3\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}\in\mathbb{Z}^2\\
&\iff 8c-6d=0\ (\bmod 12)\text{ and } -2c+3d=0\ (\bmod 12)\\
&\iff 2c=3d\ (\bmod 12)
\end{aligned}$$

For $(a,b)\in\mathbb{Z}^2$ let us write $[a,b]=(a,b)+H\in G/H$. With the help of a picture of H, or by using the formula $(c,d)\in H\iff 2c=3d\ (\bmod 12)$, we find that $\big|[1,1]\big|=12$ with $\langle[1,1]\rangle=\big\{[0,0],[1,1],[2,2],\cdots,[11,11]\big\}$ (note that $[12,12]=[0,0]$ in G/H since $(12,12)\in H$). Consequently we have $\big|[k,k]\big|=12\big/\gcd(k,12)$ for all k. We could find the order of $[a,b]$ for each of the given elements (a,b) in a similar manner, but it is more amusing to notice that every element $[a,b]$ is equal to $[k,k]$ for some $0\leq k<12$. For example, we have $[1,0]=[1,0]+[9,10]=[10,10]$ (since $[9,10]\in H$), and so $\big|[1,0]\big|=\big|[10,10]\big|=\frac{12}{\gcd(10,12)}=6$. In this way, we can find the order of each of the given elements: $\big|[0,0]\big|=1$, $\big|[1,0]\big|=\big|[10,10]\big|=6$, $\big|[2,0]\big|=\big|[8,8]\big|=3$, $\big|[3,0]\big|=\big|[6,6]\big|=2$, $\big|[4,0]\big|=\big|[4,4]\big|=3$, $\big|[5,0]\big|=\big|[2,2]\big|=6$, $\big|[0,1]\big|=\big|[3,3]\big|=4$, $\big|[1,1]\big|=12$, $\big|[2,1]\big|=\big|[11,11]\big|=12$, $\big|[3,1]\big|=\big|[9,9]\big|=4$, $\big|[4,1]\big|=\big|[7,7]\big|=12$ and $\big|[5,1]\big|=\big|[5,5]\big|=12$

4: Determine which of the following five subgroups of S_4 are normal: $\big\langle(12)\big\rangle$, $\big\langle(12)(34)\big\rangle$, $\big\langle(123)\big\rangle$, $\big\langle(1234)\big\rangle$ and $\text{stab}_{S_4}(1)$.

Solution: Notice that we have $(14)(12)(14)=(24)\notin\langle(12)\rangle$, $(14)(12)(34)(14)=(13)(24)\notin\langle(12)(34)\rangle$, $(14)(123)(14)=(234)\notin\langle(123)\rangle$, $(14)(1234)(14)=(1423)\notin\langle(1234)\rangle$, and $(14)(24)(14)=(12)\notin\text{stab}_{S_4}(1)$, and so none of these subgroups are normal.