PMATH 336 Introduction to Group Theory, Solutions to the Exercises for Chapter 6

1: For each of the following groups G, find a group of the form $\mathbb{Z}_{n_1}\times\mathbb{Z}_{n_2}\times\cdots\times\mathbb{Z}_{n_l}$ with $n_i\big|n_{i+1}$ for all i, which is isomorphic to G.

(a) $G=\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_5\times\mathbb{Z}_6\times\mathbb{Z}_8\times\mathbb{Z}_9\times\mathbb{Z}_{12}\times\mathbb{Z}_{18}\times\mathbb{Z}_{25}$.

Solution:
$$\begin{aligned} G &\cong \mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_5\times(\mathbb{Z}_2\times\mathbb{Z}_3)\times\mathbb{Z}_8\times\mathbb{Z}_9\times(\mathbb{Z}_4\times\mathbb{Z}_3)\times(\mathbb{Z}_2\times\mathbb{Z}_9)\times\mathbb{Z}_{25}\\ &\cong \mathbb{Z}_2\times\mathbb{Z}_2\times(\mathbb{Z}_2\times\mathbb{Z}_3)\times(\mathbb{Z}_4\times\mathbb{Z}_3)\times(\mathbb{Z}_4\times\mathbb{Z}_9\times\mathbb{Z}_5)\times(\mathbb{Z}_8\times\mathbb{Z}_9\times\mathbb{Z}_{25})\\ &\cong \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_6\times\mathbb{Z}_{12}\times\mathbb{Z}_{180}\times\mathbb{Z}_{1800}\end{aligned}$$

(b) $G=U_{180}$

Solution:
$$\begin{aligned} G &= U_{4\cdot5\cdot9}\\ &\cong U_4\times U_5\times U_9\\ &\cong \mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_6\\ &\cong \mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_3\\ &\cong \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{12}\end{aligned}$$

(c) $G=U(60)/\langle 29\rangle$.

Solution: In $U(60)$ we have $\langle 29\rangle=\{1,29\}$ and also $\langle 7\rangle=\{1,7,49,43\rangle$ and $\langle 59\rangle=\{1,59\}$. Now $\big|U(60)\big|=16$, so $\big|G\big|=\big|U(60)/\langle 29\rangle\big|=16/2=8$, and so G is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2\times\mathbb{Z}_4$ or to $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. Looking at the powers of 7 in $U(60)$, we see that $7\langle 29\rangle$ has order 4 in G, so G is not isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. Also, note that $49\langle 29\rangle$ and $59\langle 29\rangle$ both have order 2 in G, so G cannot be isomorphic to $\mathbb{Z}_8$. Thus $G\cong\mathbb{Z}_2\times\mathbb{Z}_4$.

2: (a) List all of the abelian groups of order 1,500.

Solution: We have $1500=2^2 3^1 5^3$, and there are 2 ways to partition 2 (namely (2) and (1,1)), 1way to partition 1, 3 ways to partition 3 (namely (3) and (1,2) and (1,1,1)), so there are $2\cdot1\cdot3=6$ abelian groups of order 1500. They are $\mathbb{Z}_4\times\mathbb{Z}_3\times\mathbb{Z}_{125}$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_{125}$, $\mathbb{Z}_4\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_{25}$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_{25}$, $\mathbb{Z}_4\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_5$ and $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_5$.

(b) Determine the number of abelian groups of order 160,000.

Solution: We have $160,000=2^8 5^4$. There are 22 ways to partition 8, that is to find $(k_1,k_2,\cdots,k_l)$ with $1\le k_1\le k_2\le\cdots\le k_l$ and $\sum k_i=8$, namely $(1,1,1,1,1,1,1,1)$, $(1,1,1,1,1,1,2)$, $(1,1,1,1,2,2)$, $(1,1,2,2,2)$, $(2,2,2,2)$, $(1,1,1,1,1,3)$, $(1,1,1,2,3)$, $(1,2,2,3)$, $(1,1,3,3)$, $(2,3,3)$, $(1,1,1,1,4)$, $(1,1,2,4)$, $(2,2,4)$, $(1,3,4)$, $(4,4)$, $(1,1,1,5)$, $(1,2,5)$, $(3,5)$, $(1,1,6)$, $(2,6)$, $(1,7)$ and (8). There are 5 ways to partition 4, namely $(1,1,1,1)$, $(1,1,2)$, $(2,2)$, $(1,3)$, (4). So there are $22\cdot5=110$ abelian groups of order 160,000.