PMATH 336 Introduction to Group Theory, Solutions to the Exercises for Chapter 7

1: Let $a=(2,6)$, $b=(3,-1)$ and $c=(4,2)$. Find the image of triangle with vertices at a, b and c under the isometry $\mathrm{R}_{(4,6),\frac{\pi}{2}}\mathrm{G}_{(4,2),x-2y+5=0}$.

Solution: $\mathrm{R}_{(4,6),\frac{\pi}{2}}\mathrm{G}_{(4,2),x-2y+5=0}(2,6)=\mathrm{R}_{(4,6),\frac{\pi}{2}}\mathrm{F}_{x-2y+5=0}(6,8)=\mathrm{R}_{(4,6),\frac{\pi}{2}}(8,4)=(6,10)$
$\mathrm{R}_{(4,6),\frac{\pi}{2}}\mathrm{G}_{(4,2),x-2y+5=0}(3,-1)=\mathrm{R}_{(4,6),\frac{\pi}{2}}\mathrm{F}_{x-2y+5=0}(7,1)=\mathrm{R}_{(4,6),\frac{\pi}{2}}(3,9)=(1,5)$
$\mathrm{R}_{(4,6),\frac{\pi}{2}}\mathrm{G}_{(4,2),x-2y+5=0}(4,2)=\mathrm{R}_{(4,6),\frac{\pi}{2}}\mathrm{F}_{x-2y+5=0}(8,4)=\mathrm{R}_{(4,6),\frac{\pi}{2}}(6,8)=(2,8)$
So the image of triangle abc is the triangle with vertices at $(6,10)$, $(1,5)$ and $(2,8)$.

2: Express the composite $\mathrm{R}_{(1,4),\frac{\pi}{2}}\mathrm{F}_{x+3y=3}$ as a single glide-reflection.

Solution: We choose two points, say $(0,1)$ and $(3,0)$, and find the images:
$\mathrm{R}_{(1,4),\frac{\pi}{2}}\mathrm{F}_{x+3y=3}(0,1)=\mathrm{R}_{(1,4),\frac{\pi}{2}}(0,1)=(4,3)$
$\mathrm{R}_{(1,4),\frac{\pi}{2}}\mathrm{F}_{x+3y=3}(3,0)=\mathrm{R}_{(1,4),\frac{\pi}{2}}(3,0)=(5,6)$
The line of the glide reflection must pass through the midpoint of $(1,0)$ and $(4,3)$, which is (2,2), and through the midpoint of (3,0) and (5,6), which is (4,3), and so the glide reflection line is the line $x-2y+2=0$. The translation vector of the glide reflection is the vector $\mathrm{F}_{x-2y+2=0}(5,6)-(3,0)=(7,2)-(3,0)=(4,2)$. Thus we have $\mathrm{R}_{(1,4),\frac{\pi}{2}}\mathrm{F}_{x+3y=3}=\mathrm{G}_{(4,2),x-2y+2=0}$

3: Find the symmetry group of each of the following subsets of $\mathbb{R}^2$.

(a) $X=\{(1,1),(5,3)\}$.

Solution: The symmetry group of X is $\mathrm{Sym}(X)=\{I,\mathrm{R}_{(3,2),\pi},\mathrm{F}_{x-2y+1=0},\mathrm{F}_{2x+y=8}\}\cong D_2$.

(b) $Y=L\cup M$ where L is the line $x+y=1$ and M is the line $x+y=3$.

Solution: This symmetry group is infinite. If we let N be the line $x+y=2$ and let V be the vector space $x+y=0$, then $\mathrm{Sym}(Y)=\{\mathrm{T}_u\big|u\in V\}\cup\{\mathrm{R}_{p,\pi}\big|p\in N\}\cup\{\mathrm{F}_K\big|K\perp V\}\cup\{\mathrm{G}_{u,N}\big|u\in V\}$.

4: Let X be the polyhedron whose 12 vertices are at $(\pm2,0,\pm2)$ and $(\pm1,\pm\sqrt{3},\pm2)$ (X is a prism whose two ends are regular hexagons). Determine whether the rotation group of X is isomorphic to $\mathbb{Z}_n$, D_n, A_4, S_4 or to A_5.

Solution: Let G be the rotation group of X. Label the faces $1,2,\cdots,8$ with the top and bottom faces labeled by 1 and 2. Then $\mathrm{orb}(1)=\{1,2\}$ so $\big|\mathrm{orb}(1)\big|=2$ and $\mathrm{stab}(1)$ consists of the rotations about the z-axis by multiples of $\frac{\pi}{6}$ so $\big|\mathrm{stab}(1)\big|=6$, and so we have $|G|=12$. Thus $G\cong\mathbb{Z}_{12}$, D_6 or A_4. Now, $\mathbb{Z}_{12}$ only has 1 element of order 2 and A_4 only has 3 elements of order 2, but G has at least 7 elements of order 2, namely the rotations by π about each of the axes $(1,0,0)$, $(\pm\sqrt{3},1,0)$, $(\pm1,\sqrt{3},0)$, $(0,1,0)$ and $(0,0,1)$. Thus $G\cong D_6$.

5: (a) How many 8-bead necklaces can be made (up to D_8 symmetry) using beads of 2 colours.

Solution: Let X be the set of all possible colourings, without considering the D_8 symmetry, so $|X| = 2^8$. Consider D_8 as a subgroup of $\mathrm{Perm}(X)$. We make a table showing the value of $\big|\mathrm{fix}(\alpha)\big|$ for each $\alpha \in D_8$.

α	#	$\big|\mathrm{fix}(\alpha)\big|$
I	1	2^8
R_4	1	2^4
R_2, R_6	2	2^2
$R_1, R_3, R_5.R_7$	4	2^1
F_0, F_2, F_4, F_6	4	2^5
F_1, F_3, F_5, F_7	4	2^4

So the number of colourings, up to the D_8 symmetry, is equal to the number of orbits which is equal to $\frac{1}{16}\big(1 \cdot 2^8 + 1 \cdot 2^4 + 2 \cdot 2^2 + 4 \cdot 2^1 + 4 \cdot 2^5 + 4 \cdot 2^4\big) = 30$.

(b) How many ways (up to rotational symmetry) can the faces of a regular octahedron be coloured using 2 colours?

Solution: Let G be the rotation group of the regular octahedron. If we consider G as a subgroup of the permutations of the faces, which we label by $1, 2, \cdots 8$, then $\big|\mathrm{orb}(1)\big| = 8$ and $\big|\mathrm{stab}(1)\big| = 3$ and so we have $|G| = 24$. Now, let X be the set of all colourings, without considering the symmetry, so that $|X| = 2^8$, and consider G as a subgroup of $\mathrm{Perm}(X)$. We make a table showing $\big|\mathrm{fix}(\alpha)\big|$ for each $\alpha \in G$.

α	#	$\big|\mathrm{fix}(\alpha)\big|$	
the identity	1	2^8	
rotation by $\pm 120°$ about an axis through a pair of opposite faces	8	2^4	(2 groups of 3, 2 groups of 1)
rotation by $180°$ about an axis through a pair of opposite edges	6	2^4	(4 groups of 2)
rotation by $\pm 90°$ about an axis through a pair of opposite vertices	6	2^2	(2 groups of 4)
rotation by $180°$ about an axis through a pair of opposite vertices	3	2^4	(4 groups of 4)

So the number of colourings, up to the rotational symmetry, is equal to the number of orbits which is equal to $\frac{1}{24}\big(1 \cdot 2^8 + 8 \cdot 2^4 + 6 \cdot 2^4 + 6 \cdot 2^2 + 3 \cdot 2^4\big) = 23$.