PMATH 347 Groups and Rings, Assignment 1 Due Tues May 21, 11:00 pm

1: Determine which of the following are groups, and which of the groups are abelian.

(a) $\mathbb{R}^*$ under division.

(b) The set of all subsets of $\{1, 2, 3, 4\}$ under union.

(c) $\{x \in \mathbb{R} \,|\, x^2 \in \mathbb{Z}\}$ under addition.

(d) $\{(a, b) \in \mathbb{R}^2 \,|\, b \neq 0\}$ under the operation $*$ defined by $(a, b) * (c, d) = (c + ad, bd)$.

2: (a) Let G be a group. Suppose that for all $a, b, c, d, x \in G$, if $axb = cxd$ then $ab = cd$. Show that G is abelian.

(b) Let G be a finite group. Show that there are an odd number of elements $x \in G$ with $x^3 = e$.

(c) Let G be a non-empty finite set with a binary operation $* : G \times G \to G$ with the following properties:
(1) associativity: for all $a, b, c \in G$ we have $(a * b) * c = a * (b * c)$,
(2) right cancellation: for all $a, b, c \in G$, if $a * c = b * c$ then $a = b$, and
(3) left cancellation: for all $a, b, c \in G$, if $c * a = c * b$ then $a = b$.
Show that G is group under $*$.

3: Let R be a ring with 1.

(a) Let $a, b \in R$. Suppose that $a^3 = a$ and $ab + ba = 1$. Show that $a^2 = 1$.

(b) Let $a, b \in R$. Suppose that a and b and $a + b$ are units. Show that $a^{-1} + b^{-1}$ is a unit.

(c) Show that if $a^2 = a$ for all $a \in R$ then R is commutative.

4: (a) Find $|GL_2(\mathbb{Z}_4)|$.

(b) List every element in each conjugacy class in $GL_2(\mathbb{Z}_2)$.

(c) Find the number of elements in the conjugacy class of $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ in $GL_2(\mathbb{Z}_3)$.