PMATH 347 Groups and Rings, Assignment 3 Due Tues June 18, 11:00 pm

**1:** For $n \in \mathbb{Z}^+$, let $\mathbb{Z}_n[i] = \{a + ib \,|\, a, b \in \mathbb{Z}_n\}$, with addition and multiplication defined in the obvious way by $(a+ib)+(c+id) = (a+c)+i(b+d)$ and $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$. You may assume, without proof, that $\mathbb{Z}_n[i]$ is a ring.

(a) Find all the units and all the zero divisors in the ring $\mathbb{Z}_4[i]$.

(b) Without proof, list all of the subrings of $\mathbb{Z}_4[i]$.

(c) Find (with proof) all primes $p$ with $p < 12$ such that $\mathbb{Z}_p[i]$ is a field.

**2:** (a) Consider the ring $\mathcal{C}^0(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ under addition and mutiplication. Prove that the units in $\mathcal{C}^0(\mathbb{R})$ are the nowhere zero functions, and the zero-divisors in $\mathcal{C}^0(\mathbb{R})$ are the functions which are not identically zero, but which are zero in some open interval.

(b) Let $F$ be a field and consider the ring $F\big[[x]\big]$ of formal power series in $x$. Find all the units and all the zero divisors in $F\big[[x]\big]$.

(c) Consider the group $S_\infty = \text{Perm}(\mathbb{Z}^+)$ of bijective maps $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$ under composition. Let $H$ be the set of all elements of finite order in $S_\infty$. Determine whether $H \leq S_\infty$.

**3:** (a) In $S_9$, let $\alpha = (1548)(2936)$ and $\beta = (16574)(38)$. Find $(-1)^{\alpha\beta}$ and $|\alpha\beta|$.

(b) In $S_8$, let $\beta = (123)(456)$. Find every element $\alpha \in S_8$ such that $\alpha^2 = \beta$.

(c) In $S_{10}$, let $\beta = (123)(456)(78)$. Find the number of elements $\alpha \in S_{10}$ such that $\alpha\beta = \beta\alpha$.

**4:** (a) Find the number of cyclic subgroups of $A_6$.

(b) For $n \in \mathbb{Z}^+$, let $P(n)$ be the probability that when one of the $(2n)!$ elements $\sigma \in S_{2n}$ is selected at random and written using cycle notation, one of the cycles has length $\ell > n$. Find $\lim\limits_{n\to\infty} P(n)$.

Hint: express $P(n)$ as a sum over $n < \ell \leq 2n$, and interpret the sum as a Riemann sum.