PMATH 347 Groups and Rings, Exercises for Chapter 10

1: In each case, determine whether A is an ideal of the ring R.

(a) $R = \mathbb{Z} \times \mathbb{Z}$, $A = \{(k,k) | k \in \mathbb{Z}\}$.

(b) $R = \text{Func}(\mathbb{R}, \mathbb{R})$, $A = \left\{ f : \mathbb{R} \to \mathbb{R} \middle| \int_0^1 f(x)\,dx = 0 \right\}$.

(c) $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}$, $A = \left\{ \begin{pmatrix} a & 2b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}$.

2: For each of the following quotient rings, list the elements, construct the multiplication table, and determine whether the quotient ring is a field.

(a) $3\mathbb{Z}/\langle 12 \rangle$

(b) $\mathbb{Z}_2[x]/\langle x^2 + x + 1 \rangle$

3: Determine the number of elements in the ring $\mathbb{Z}[i]/\langle 2 - 2i \rangle$.

4: Let A and B be ideals in a ring R. One can show that $A \cap B$, $A + B = \{a + b | a \in A, b \in B\}$, and $AB = \{a_1 b_1 + \cdots + a_n b_n | a_i \in A, b_i \in B\}$ are ideals of R (you do *not* need to show this). If $R = \mathbb{Z}$, $A = \langle 12 \rangle$ and $B = \langle 30 \rangle$ then find $A \cap B$, $A + B$ and AB.

5: In the ring $\mathbb{Z}[x]$, show that the ideal $\langle x \rangle$ is prime but not maximal.

6: (a) Find all the ring homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_2 \times \mathbb{Z}_6$.

(b) Find all the ring homomorphisms from $\mathbb{Z}_2 \times \mathbb{Z}_6$ to $\mathbb{Z}_{12}$.

7: For each of the following pairs of rings R, and S, determine whether $R \cong S$.

(a) $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, $S = \mathbb{Z}_2[i]$

(b) $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, $S = \mathbb{Z}_2[x]/\langle x^2 + x \rangle$