PMATH 347 Groups and Rings, Solutions to the Exercises for Chapter 10

1: In each case, determine whether A is an ideal of the ring R.

(a) $R = \mathbb{Z} \times \mathbb{Z}$, $A = \{(k,k) | k \in \mathbb{Z}\}$.

Solution: A is a subring but not an ideal, since it is not closed under multiplication by elements of R. For example, $(1,1) \in A$ and $(1,2) \in R$ but $(1,1)(1,2) = (1,2) \notin A$.

(b) $R = \text{Func}(\mathbb{R}, \mathbb{R})$, $A = \left\{ f : \mathbb{R} \to \mathbb{R} \middle| \int_0^1 f(x)\,dx = 0 \right\}$.

Solution: A is not even a subring. For example if $f(x) = 2x - 1$ then $f \in A$ but $f^2 \notin A$.

(c) $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}$, $A = \left\{ \begin{pmatrix} a & 2b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}$.

Solution: A is a subring but not an ideal. For example $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$ and $\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \in A$ but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin A$.

2: For each of the following quotient rings, list the elements, construct the multiplication table, and determine whether the quotient ring is a field.

(a) $3\mathbb{Z}/\langle 12 \rangle$

Solution: $3\mathbb{Z}/\langle 12 \rangle = \{[0], [3], [6], [9]\}$, which we write as $\{0, 3, 6, 9\}$. The multiplication table is

	0	3	6	9
0	0	0	0	0
3	0	9	6	3
6	0	6	0	6
9	0	3	6	9

Since 6 is a zero divisor, $3\mathbb{Z}/\langle 12 \rangle$ is not an integral domain (and hence not a field). (The ring does have an identity, namely 9).

(b) $\mathbb{Z}_2[x]/\langle x^2 + x + 1 \rangle$

Solution: $\mathbb{Z}_2[x]/\langle x^2 + x + 1 \rangle = \{[0], [1], [x], [1 + x]\}$, which we write as $\{0, 1, x, 1 + x\}$. The multiplication table is

	0	1	x	$1+x$
0	0	0	0	0
1	0	1	x	$1+x$
x	0	x	$1+x$	1
$1+x$	0	$1+x$	1	x

This ring is commutative and has a 1, and every element has an inverse, so its a field.

3: Determine the number of elements in the ring $\mathbb{Z}[i]/\langle 2-2i\rangle$.

Solution: $\mathbb{Z}[i]/\langle 2-2i\rangle$ has 8 elements. The easiest way to see this is to draw a picture. It can also be shown as follows. Since $[2-2i]=[0]$ we have $[2i]=[2]$ and so $[2ki]=[2k]$ and $[(2k+1)i]=[2k+i]$. So we have

$$[a+bi]=\begin{cases}[a+b] & \text{, if } b \text{ is even}\\ [(a+b-1)+i] & \text{, if } b \text{ is odd}\end{cases}$$

Thus the elements of $\mathbb{Z}[i]/\langle 2-2i\rangle$ are all of the form $[c]$ or $[c+i]$ for some $c\in\mathbb{Z}$. Also, we have $[2+2i]=[2-2i][i]=[0][i]=[0]$, hence $[4]=[2+2i]+[2-2i]=[0]+[0]=[0]$, and so $[c]=[c+4k]$ for all $k\in\mathbb{Z}$. Thus $\mathbb{Z}[i]/\langle 2-2i\rangle=\{[0],[1],[2],[3],[i],[1+i],[2+i],[3+i]\}$. It remains to show that these 8 elements are all distinct. To do this, let $c,d\in\mathbb{Z}$. We have $[c]=[d] \iff c-d\in\langle 2-2i\rangle \iff c-d=(2-2i)(x+iy)=2(x+y)+2(y-x)i$ for some $x,y\in\mathbb{Z} \iff 2(y-x)=0$ and $2(x+y)=c-d$ for some $x,y\in\mathbb{Z} \iff y=x$ and $4x=c-d$ for some $x,y\in\mathbb{Z} \iff c=d$ modulo 4. This shows that the elements $[0]$, $[1]$, $[2]$ and $[3]$ are distinct. Also, we have $[c+i]=[d] \iff (c+i)-d\in\langle 2-2i\rangle \iff (c-d)+i=(2-2i)(x+iy)=2(x+y)+2(y-x)i$ for some $x,y\in\mathbb{Z} \iff 2(y-x)=1$ and $2(x+y)=c-d$ for some $x,y\in\mathbb{Z}$, which never occurs, since $2(y-x)$ is even so $2(y-x)\neq 1$. This shows that the elements of the form $[c]$ are all distinct from the elements of the form $[d+i]$. Finally, we have $[c+i]=[d+i] \iff [c]=[d] \iff c=d$ modulo 4. This shows that the elements $[i]$, $[1+i]$, $[2+i]$ and $[3+i]$ are all distinct.

4: Let A and B be ideals in a ring R. One can show that $A\cap B$, $A+B=\{a+b|a\in A, b\in B\}$, and $AB=\{a_1b_1+\cdots+a_nb_n|a_i\in A, b_i\in B\}$ are ideals of R (you do *not* need to show this). If $R=\mathbb{Z}$, $A=\langle 12\rangle$ and $B=\langle 30\rangle$ then find $A\cap B$, $A+B$ and AB.

Solution: $x\in\langle k\rangle\cap\langle l\rangle \iff k\big|x$ and $l\big|x \iff \text{lcm}(k,l)\big|x$, so $\langle k\rangle\cap\langle l\rangle=\langle\text{lcm}(k,l)\rangle$. In particular $\langle 12\rangle\cap\langle 30\rangle=\langle 60\rangle$.

Also, $x\in\langle k\rangle+\langle l\rangle \iff x=ks+lt$ for somr $s,t\in\mathbb{Z} \iff \gcd(k,l)\big|x$ (by first year algebra), and so $\langle k\rangle+\langle l\rangle=\langle\gcd(k,l)\rangle$. In particular, $\langle 12\rangle+\langle 30\rangle=\langle 6\rangle$.

Finally, we have $x\in\langle k\rangle\langle l\rangle \iff x=a_1b_1+\cdots+a_nb_n$ for some $n\in\mathbb{Z}$, $a_i\in\langle k\rangle$ and $b_i\in\langle l\rangle \iff x=(k\,s_1)(l\,t_1)+\cdots+(k\,s_n)(l\,t_n)$ for some $n,s_i,t_i\in\mathbb{Z} \iff x=r_1kl+\cdots+r_nkl$ for some $n,r_i\in\mathbb{Z} \iff x=r\,kl$ for some $r\in\mathbb{Z}$, and so $\langle k\rangle\langle l\rangle=\langle kl\rangle$. In particular $\langle 12\rangle\langle 30\rangle=\langle 360\rangle$.

5: In the ring $\mathbb{Z}[x]$, show that the ideal $\langle x\rangle$ is prime but not maximal.

Solution: Note first that $\langle x\rangle=\big\{f\in\mathbb{Z}[x]\big|f(0)=0\big\}$. The ideal $\langle x\rangle$ is prime, because for $f,g\in\mathbb{Z}[x]$ we have $fg\in\langle x\rangle \Longrightarrow (fg)(0)=0 \Longrightarrow f(0)g(0)=0 \Longrightarrow f(0)=0$ or $g(0)=0 \Longrightarrow f\in\langle x\rangle$ or $g\in\langle x\rangle$. On the other hand, $\langle x\rangle$ is not maximal since for any integer $n\geq 2$, the set $A_n=\big\{f\in\mathbb{Z}[x]\big|f(0)\in n\mathbb{Z}\big\}$ is an ideal of $\mathbb{Z}[x]$ which properly contains $\langle x\rangle$.

6: (a) Find all the ring homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_2\times\mathbb{Z}_6$.

Solution: We know that the ring homomorphisms are of the form $\phi(k)=(ka,kb)$, where $(a,b)\in\mathbb{Z}_2\times\mathbb{Z}_6$ with $(a^2,b^2)=(a,b)$ and $(12a,12b)=0$. In $\mathbb{Z}_2\times\mathbb{Z}_6$, we always have $(12a,12b)=0$. In $\mathbb{Z}_2$ we have $0^2=0$ and $1^2=1$. In $\mathbb{Z}_6$ we have $0^2=0$, $1^2=1$, $2^2=4\neq 2$, $3^2=3$, $4^2=4$, and $5^2=1\neq 5$. Thus there are 8 ring homomorphisms $\phi:\mathbb{Z}_{12}\to\mathbb{Z}_2\times\mathbb{Z}_6$, namely $\phi(k)=(ka,kb)$, where $a=0,1$ and $b=0,1,3,4$.

(b) Find all the ring homomorphisms from $\mathbb{Z}_2\times\mathbb{Z}_6$ to $\mathbb{Z}_{12}$.

Solution: Suppose that $\phi:\mathbb{Z}_m\times\mathbb{Z}_n\to R$ is a ring homomorphism, where $m,n\in\mathbb{Z}$ and R is any ring. Say $\phi(1,0)=a$ and $\phi(0,1)=b$. Then ϕ is given by $\phi(k,l)=\phi\big(k(1,0)+l(0,1)\big)=k\phi(1,0)+l\phi(0,1)=ka+lb$. Note that we must have $ma=m\phi(1,0)=\phi\big(m(1,0)\big)=\phi(0,0)=0$, and similarly $nb=0$. Also we must have $a^2=\big(\phi(1,0)\big)^2=\phi\big((1,0)^2\big)=\phi(1,0)=a$, and similarly we must have $b^2=b$. Thirdly, we must have $ab=\phi(1,0)\phi(0,1)=\phi\big((1,0)(0,1)\big)=\phi(0,0)=0$, and similarly we must have $ba=0$. Conversely, check that if $ma=0$ and $nb=0$ then the map $\phi(k.l)=ka+lb$ is well defined and preserves addition, and that if $a^2=a$ and $b^2=b$ and $ab=ba=0$ then the map ϕ preserves multiplication. In particular, the ring homomorphisms $\phi:\mathbb{Z}_2\times\mathbb{Z}_6\to\mathbb{Z}_{12}$ are given by $\phi(k,l)=ka+lb$, where $a,b\in\mathbb{Z}_{12}$ satisfy $2a=0$, $6b=0$, $a^2=a$, $b^2=b$, and $ab=ba=0$. Check that $a=0$ and $b=0$ or 4, so there are two ring homomorphisms.

7: For each of the following pairs of rings R, and S, determine whether $R \cong S$.

(a) $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, $S = \mathbb{Z}_2[i]$

Solution: Note first that for any rings R and S, if $\phi : R \to S$ is a ring isomorphism, then we have $ab = 0 \in R \iff \phi(a)\phi(b) = 0 \in S$, $ab = 1 \in R \iff \phi(a)\phi(b) = 1 \in S$ and also $a^2 = a \in R \iff \phi(a)^2 = \phi(a) \in S$. This shows that if $R \cong S$ then R and S must have the same number of zero divisors, units, and idempotents. Check that $Z_2 \times \mathbb{Z}_2$ has 2 zero divisors (namely (1,0) and (0,1)), 1 unit (namely (1,1)) and 4 idempotents. Check, on the other hand, that $\mathbb{Z}_2[i]$ has 1 zero divisor (namely $1 + i$), 2 units (namely 1 and i), and 2 idempotents (namely 0 and 1). Thus $\mathbb{Z}_2 \times \mathbb{Z}_2$ is *not* isomorphic to $\mathbb{Z}_2[i]$.

(b) $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, $S = \mathbb{Z}_2[x]/\langle x^2 + x\rangle$

Solution: Define $\phi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}[x]/\langle x^2 + x\rangle$ by $\phi(0, 0) = 0$, $\phi(1, 0) = x$, $\phi(0, 1) = 1 + x$ and $\phi(1, 1) = 1$. Check that ϕ is an isomorphism by writing out the addition and multiplication tables for R and S.