PMATH 347 Groups and Rings, Solutions to the Exercises for Chapter 6

1: For each of the following groups G, find a group of the form $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_l}$ with $n_i \big| n_{i+1}$ for all i, which is isomorphic to G.

(a) $G = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_6 \times \mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_{12} \times \mathbb{Z}_{18} \times \mathbb{Z}_{25}$.

Solution:
$$\begin{aligned} G &\cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times (\mathbb{Z}_2 \times \mathbb{Z}_3) \times \mathbb{Z}_8 \times \mathbb{Z}_9 \times (\mathbb{Z}_4 \times \mathbb{Z}_3) \times (\mathbb{Z}_2 \times \mathbb{Z}_9) \times \mathbb{Z}_{25} \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_3) \times (\mathbb{Z}_4 \times \mathbb{Z}_3) \times (\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5) \times (\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}) \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{12} \times \mathbb{Z}_{180} \times \mathbb{Z}_{1800} \end{aligned}$$

(b) $G = U_{180}$

Solution:
$$\begin{aligned} G &= U_{4\cdot 5\cdot 9} \\ &\cong U_4 \times U_5 \times U_9 \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6 \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12} \end{aligned}$$

(c) $G = U(60)/\langle 29\rangle$.

Solution: In $U(60)$ we have $\langle 29\rangle = \{1, 29\}$ and also $\langle 7\rangle = \{1, 7, 49, 43\}$ and $\langle 59\rangle = \{1, 59\}$. Now $\big|U(60)\big| = 16$, so $\big|G\big| = \big|U(60)/\langle 29\rangle\big| = 16/2 = 8$, and so G is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Looking at the powers of 7 in $U(60)$, we see that $7\langle 29\rangle$ has order 4 in G, so G is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Also, note that $49\langle 29\rangle$ and $59\langle 29\rangle$ both have order 2 in G, so G cannot be isomorphic to $\mathbb{Z}_8$. Thus $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

2: (a) List all of the abelian groups of order 1,500.

Solution: We have $1500 = 2^2 3^1 5^3$, and there are 2 ways to partition 2 (namely (2) and (1,1)), 1way to partition 1, 3 ways to partition 3 (namely (3) and (1,2) and (1,1,1)), so there are $2 \cdot 1 \cdot 3 = 6$ abelian groups of order 1500. They are $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_{125}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{125}$, $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{25}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{25}$, $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$.

(b) Determine the number of abelian groups of order 160,000.

Solution: We have $160,000 = 2^8 5^4$. There are 22 ways to partition 8, that is to find $(k_1, k_2, \cdots, k_l)$ with $1 \le k_1 \le k_2 \le \cdots \le k_l$ and $\sum k_i = 8$, namely $(1,1,1,1,1,1,1,1)$, $(1,1,1,1,1,1,2)$, $(1,1,1,1,2,2)$, $(1,1,2,2,2)$, $(2,2,2,2)$, $(1,1,1,1,1,3)$, $(1,1,1,2,3)$, $(1,2,2,3)$, $(1,1,3,3)$, $(2,3,3)$, $(1,1,1,1,4)$, $(1,1,2,4)$, $(2,2,4)$, $(1,3,4)$, $(4,4)$, $(1,1,1,5)$, $(1,2,5)$, $(3,5)$, $(1,1,6)$, $(2,6)$, $(1,7)$ and (8). There are 5 ways to partition 4, namely $(1,1,1,1)$, $(1,1,2)$, $(2,2)$, $(1,3)$, (4). So there are $22 \cdot 5 = 110$ abelian groups of order 160,000.