PMATH 347 Groups and Rings, Exercises for Chapter 9

1: For each of the rings $\mathbb{Z}_6$, $\mathbb{Z}_2[i]$ and $\text{Func}(\mathbb{Z}_2, \mathbb{Z}_2)$, do the following:

(a) Make a multiplication table.

(b) Find all the zero divisors.

(c) Find all the units.

(d) Determine whether the ring is a field, an integral domain, or neither.

2: Determine which of the following are rings; for each ring, determine whether it has an identity.

(a) $Q = \left\{ \begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix} \middle| a, b \in \mathbb{Z} \right\} \subset M_2(\mathbb{Z})$

(b) $R = \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} \middle| a, b \in \mathbb{Z} \right\} \subset M_2(\mathbb{Z})$

(c) $S = \left\{ f : \mathbb{R} \to \mathbb{R} \right\}$ under the operations addition $+$ and composition $\circ$

(d) $T = \left\{ \frac{a}{b} \middle| a, b \in \mathbb{Z} \text{ with } a \text{ even and } b \text{ odd} \right\}$

3: Find the smallest subring of $\mathbb{Q}$ which contains $\frac{2}{3}$.

4: An element a in a ring R is called **nilpotent** if $a^n = 0$ for some $n \in \mathbb{Z}$. An element a is called an **idempotent** if $a^2 = a$. Find all the zero divisors, all the units, all the nilpotent elements, and all the idempotents in the ring $R = \mathbb{Z}_3 \oplus \mathbb{Z}_6$.

5: Find all the solutions of $x^2 - x + 2 = 0$, where

(a) $x \in \mathbb{Z}_8$

(b) $x \in \mathbb{Z}_3[i]$.

6: Find all solutions to $X^2 = I$, where

(a) $X \in M_2(\mathbb{R})$

(b) $X \in M_2(\mathbb{Z}_2)$.

7: (a) If $ab = a$ and $ba = b$ in a ring, show that $a^2 = a$.

(b) If $ab + ba = 1$ and $a^3 = a$ in a ring, show that $a^2 = 1$.