PMATH 347 Groups and Rings, Solutions to the Exercises for Chapter 9

1: For each of the rings $\mathbb{Z}_6$, $\mathbb{Z}_2[i]$ and $\text{Func}(\mathbb{Z}_2,\mathbb{Z}_2)$, do the following:

(a) Make a multiplication table.

Solution: Let us write $\text{Func}(\mathbb{Z}_2,\mathbb{Z}_2)=\{e,\ell,f,g\}$ where $e(x)=0$, $\ell(x)=1$, $f(x)=x$, and $g(x)=1+x$. The multiplication tables are as follows:

$\mathbb{Z}_6$

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

$\mathbb{Z}_2[i]$

	0	1	i	$1+i$
0	0	0	0	0
1	0	1	i	$1+i$
i	0	i	1	$1+i$
$1+i$	0	$1+i$	$1+i$	0

$\text{Func}(\mathbb{Z}_2,\mathbb{Z}_2)$

	e	ℓ	f	g
e	e	e	e	e
ℓ	e	ℓ	f	g
f	e	f	f	e
g	e	g	e	g

(b) Find all the zero divisors.

Solution: In $\mathbb{Z}_6$ the zero divisors are 2, 3 and 4. In $\mathbb{Z}_2[i]$ the only zero divisor is $1+i$, and in $\text{Func}(\mathbb{Z}_2,\mathbb{Z}_2)$ the zero divisors are f and g.

(c) Find all the units.

Solution: In $\mathbb{Z}_6$ are 1 and 5; in $\mathbb{Z}_2[i]$ the units are 1 and i; and in $\text{Func}(\mathbb{Z}_2,\mathbb{Z}_2)$ the only unit is ℓ.

(d) Determine whether the ring is a field, an integral domain, or neither.

Solution: All three rings have zero-divisors, so none of them are integral domains (or fields).

2: Determine which of the following are rings; for each ring, determine whether it has an identity.

(a) $Q=\left\{\begin{pmatrix} a & a+b \\ a+b & b\end{pmatrix}\middle|\, a,b\in\mathbb{Z}\right\}\subset M_2(\mathbb{Z})$

Solution: Q is not a ring since it is not closed under multiplication. Its easy to find a counterexample.

(b) $R=\left\{\begin{pmatrix} a & a \\ b & b\end{pmatrix}\middle|\, a,b\in\mathbb{Z}\right\}\subset M_2(\mathbb{Z})$

Solution: R is a subring of $M_2(\mathbb{Z})$, since $0\in R$ and R is closed under addition, subtraction and multiplication. To see that R is closed under multiplication, note that $\begin{pmatrix} a & a \\ b & b\end{pmatrix}\begin{pmatrix} c & c \\ d & d\end{pmatrix}=\begin{pmatrix} e & e \\ f & f\end{pmatrix}$, where $e=a(c+d)$ and $f=b(c+d)$. R does *not* have an identity, since if $\begin{pmatrix} a & a \\ b & b\end{pmatrix}$ were the identity then we would need to have $a(c+d)=c$ for all c and d, and no such $a\in\mathbb{Z}$ exists.

(c) $S=\Big\{f:\mathbb{R}\to\mathbb{R}\Big\}$ under the operations addition $+$ and composition $\circ$

Solution: S is not a ring since it is not true that $f\circ(g+h)=f\circ g+f\circ h$ for all f,g,h. For example, if f is the constant function $f=1$, then $f\circ(g+h)=1$ but $f\circ g+f\circ h=2$.

(d) $T=\left\{\frac{a}{b}\middle|\, a,b\in\mathbb{Z} \text{ with } a \text{ even and } b \text{ odd}\right\}$

Solution: T is a ring. We have $0\in T$ since $0=\frac{0}{3}$. Also T is closed under $+$, $-$ and $\cdot$ since if a and c are even , and b and d are odd, then we have $\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd}$ and $\frac{a}{b}\frac{c}{d}=\frac{ac}{bd}$; the numerators are $ad\pm bc$ and ac, which are even, and the denominators are both bd, which is odd.

3: Find the smallest subring of $\mathbb{Q}$ which contains $\frac{2}{3}$.

Solution: Let S be the smallest such subring. We have $\dfrac{2}{3} = \dfrac{6}{9} \in S$ and $\dfrac{2}{3}\dfrac{2}{3} = \dfrac{4}{9} \in S$ and so $\dfrac{6}{9} - \dfrac{4}{9} = \dfrac{2}{9} \in S$. Suppose (inductively) that $\dfrac{2}{3^n} \in S$ for some fixed $n \in \mathbb{Z}_+$. Then $\dfrac{2}{3^n} = \dfrac{6}{3^{n+1}} \in S$ and $\dfrac{2}{3^n}\dfrac{2}{3} = \dfrac{4}{3^{n+1}} \in S$ and so $\dfrac{6}{3^{n+1}} - \dfrac{4}{3^{n+1}} = \dfrac{2}{3^{n+1}} \in S$. By induction, $\dfrac{2}{3^n} \in S$ for all $n \in \mathbb{Z}_+$. Thus $T = \left\{ \dfrac{2k}{3^n} \middle| k \in \mathbb{Z}, n \in \mathbb{Z}_+ \right\} \subset S$. It is easy to check that T is a ring, and so we must have $S = T$.

4: An element a in a ring R is called **nilpotent** if $a^n = 0$ for some $n \in \mathbb{Z}$. An element a is called an **idempotent** if $a^2 = a$. Find all the zero divisors, all the units, all the nilpotent elements, and all the idempotents in the ring $R = \mathbb{Z}_3 \oplus \mathbb{Z}_6$.

Solution: The zero-divisors are the elements (a, b) with $a = 0$, 1 or 2 in $\mathbb{Z}_3$ and $b = 2$, 3 or 4 in $\mathbb{Z}_6$ (there are 9 zero-divisors). The units are the elements (a, b) with $a = 1$ or 2 and $b = 1$ or 5 (there are 4 units). The nilpotents are $(0, 0)$ and $(0, 2)$. The idempotents are the elements (a, b) with $a = 0$ or 1 and $b = 0$, 1, 3 or 4 (there are 8 idempotents).

5: Find all the solutions of $x^2 - x + 2 = 0$, where

(a) $x \in \mathbb{Z}_8$

Solution: By trying every element, we find the solutions $x = 3$ or 6.

(b) $x \in \mathbb{Z}_3[i]$.

Solution: By trying every element, we find the solutions $x = 2 + i$ or $2 + 2i$.

6: Find all solutions to $X^2 = I$, where

(a) $X \in M_2(\mathbb{R})$

Solution: Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $X^2 = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{pmatrix}$. To get $X^2 = I$ we need $a^2 + bc = 1$ (1), $b(a+d) = 0$ (2), $c(a+d) = 0$ (3) and $d^2 + bc = 1$ (4). From equations (1) and (4) we see that $a^2 = d^2 = 1 - bc$ so $a = \pm d$. Case 1: if $a = d \neq 0$ then equations (2) and (3) imply that $b = c = 0$, and then equation (1) implies that $a = \pm 1$ so we obtain $X = \pm I$. Case 2: if $a = -d$ (this includes the case that $a = d = 0$) then equations (2) and (3) place no restriction on b and c, and then equation (1) implies that $a = \pm\sqrt{1 - bc}$ so we obtain $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where b and c may be any real numbers satisfying $bc < 1$, and where $a = \pm\sqrt{1 - bc}$ and $d = -a$. .

(b) $X \in M_2(\mathbb{Z}_2)$.

Solution: As above we let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and as above, the same 4 equations must be satisfied. However, in $\mathbb{Z}_2$ we have $x^2 = x$ for all x, so equations (1) and (4) imply that $a = d = 1 - bc$. Since $a = d$, we have $a + d = 2a = 0$ so equations (2) and (3) give no further restrictions. Thus $X = \begin{pmatrix} 1 - bc & b \\ c & 1 - bc \end{pmatrix}$ where b and c are arbitrary. To be explicit, $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

7: (a) If $ab = a$ and $ba = b$ in a ring, show that $a^2 = a$.

Solution: Suppose $ab = a$ and $ba = b$. Then $a = ab = a(ba) = (ab)a = aa = a^2$.

(b) If $ab + ba = 1$ and $a^3 = a$ in a ring, show that $a^2 = 1$.

Solution: Suppose that $ab + ba = 1$ and $a^3 = a$. Then $a^2 = a^2 + a^2 - a^2 = a^2(ab + ba) + (ab + ba)a^2 - a^2 = a^3b + a^2ba + aba^2 + ba^3 - a^2 = ab + a^2ba + aba^2 + ba - a^2 = (ab + ba) + a(ab + ba)a - a^2 = 1 + a^2 - a^2 = 1$