SYDE Advanced Math 2, Assignment 1 — Due Fri May 17, 11:00 pm

1: Consider the IVP $y' = x - y^2$ with $y(0) = 0$.

(a) Sketch the direction field for the given DE for $-2 \le x \le 3$ and $-2 \le y \le 2$ and, on the same grid, sketch the solution curve to the given IVP.

(b) Using a calculator, apply Euler's method with step size $\Delta x = 0.5$ to approximate the value of $f(3)$ where $y = f(x)$ is the solution to the given IVP.

2: (a) The substitution $u(x) = y'(x)$ and $u'(x) = y''(x)$ transforms a second order DE of the form $y'' = F(y', x)$ for $y = y(x)$ to the first order DE $u' = F(u, x)$ for $u = u(x)$. Use this substitution to solve the IVP $y'' + x(y')^2 = 0$ with $y(0) = 2$ and $y'(0) = \frac{1}{2}$.

(b) The substitution $u(y(x)) = y'(x)$ and $u'(y(x))y'(x) = y''(x)$ transforms a second order DE of the form $y'' = F(y', y)$ for $y = y(x)$ to the first order DE $u\,u' = F(u, y)$ for $u = u(y)$. Use this substitution to solve $y'' + (y')^2 = 2e^{-y}$ with $y(0) = 0$ and $y'(0) = 2$.

3: Given one solution $y = y_1(x)$ to the linear homogeneous DE $y'' + p(x) + q(x)y = 0$, we can often find a second independent solution by trying $y_2(x) = y_1(x)u(x)$ for some function $u = u(x)$. Use this method, known as **reduction of order**, to solve each of the following.

(a) Solve the DE $x^3y'' + xy' - y = 0$, given that $y = x$ is one solution.

(b) Solve the IVP $x^2y'' + 3xy' + y = 0$ with $y(1) = 2$, $y'(1) = 3$ given that $y = \frac{1}{x}$ is one solution to the DE.

4: Given two independent solutions $y = y_1(x)$ and $y = y_2(x)$ to the linear homogeneous DE

$$y'' + p(x)y' + q(x)y = 0$$

we can often find a particular solution $y = y_p(x)$ to the associated non-homogeneous DE

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

by trying $y_p(x) = y_1(x)u_1(x) + y_2(x)u_2(x)$ for some functions $u_1(x)$ and $u_2(x)$ satisfying the condition ${y_1u_1}' + {y_2u_2}' = 0$ (1). Putting $y = y_1u_1 + y_2u_2$ into the non-homogeneous DE, and using condition (1) along with the fact that y_1 and y_2 are solutions to the homogeneous DE, gives ${y_1}'{u_1}' + {y_2}'{u_2}' = r$ (2). Solving the two equations (1) and (2) allows us to find the unknown functions u_1 and u_2, and hence the particular solution $y_p = y_1u_1 + y_2u_2$. Use this method, known as **variation of parameters**, to solve each of the following.

(a) Solve the DE $x^2y'' - x(x+2)y' + (x+2)y = 2x^3$ given that $y = x$ and $y = xe^x$ are solutions to the associated homogeneous DE.

(b) Solve the DE $xy'' - (1+x)y' + y = x^2e^{2x}$ given that $y = 1 + x$ and $y = e^x$ are solutions to the associated homogeneous DE.