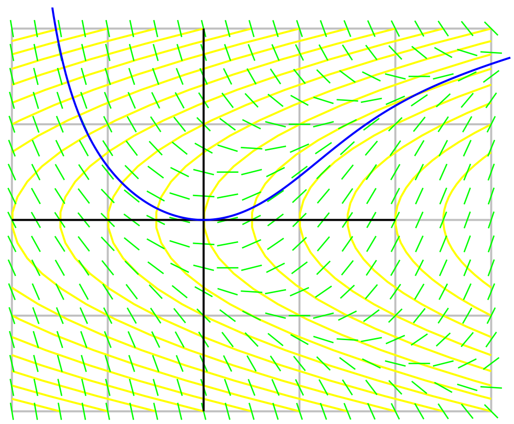
SYDE Advanced Math 2, Solutions to Assignment 1

1: Consider the IVP $y' = x - y^2$ with $y(0) = 0$.

(a) Sketch the direction field for the given DE for $-2 \le x \le 3$ and $-2 \le y \le 2$ and, on the same grid, sketch the solution curve to the given IVP.

Solution: The isocline (curve of constant slope) $y' = m$ is the sideways parabola $m = x - y^2$, or $x = y^2 + m$. The isoclines are shown in yellow, the slope field is shown in green, and the solution curve with $y(0) = 0$ is shown in blue.



(b) Using a calculator, apply Euler's method with step size $\Delta x = 0.5$ to approximate the value of $f(3)$ where $y = f(x)$ is the solution to the given IVP.

Solution: We let $x_0 = 0$ and $y_0 = 0$, then for $k \ge 0$ we set $x_{k+1} = x_k + \Delta x$ and $y_{k+1} = y_k + F(x_k, y_k)\Delta x$, where $F(x, y) = x - y^2$. We make a table listing the values of x_k, y_k and $F(x_k, y_k)$.

k	x_k	y_k	$F(x_k, y_k) = x_k - {y_k}^2$
0	0	0	0
1	0.5	0	0.5
2	1.0	0.25	0.9375
3	1.5	0.71875	0.9833984375
4	2.0	1.210449219	0.534812688
5	2.5	1.477855563	0.315942935
6	3.0	1.635827030	

Thus we have $f(3) \cong y_6 \cong 1.6$.

2: (a) Use the substitution $u(x) = y'(x)$, so that $u'(x) = y''(x)$, to solve the IVP $y'' + x(y')^2 = 0$ with $y(0) = 2$ and $y'(0) = \frac{1}{2}$.

Solution: Make the substitution $y' = u$, $y'' = u'$. The DE becomes $u' + xu^2 = 0$. This is separable since we can write it as $-\frac{1}{u^2}u' = x$. Integrate both sides (with respect to x) to get $\frac{1}{u} = \frac{1}{2}x^2 + a$. Put in $x = 0$, $u = y' = \frac{1}{2}$ to get $2 = a$, so we have $\frac{1}{u} = \frac{1}{2}x^2 + 2 = \frac{x^2+4}{2}$, that is $y' = u = \frac{2}{x^2+4}$. Integrate to get $y = \displaystyle\int \frac{2\,dx}{x^2+4} = \tan^{-1}\left(\frac{x}{2}\right) + b$. Put in $x = 0$, $y = 2$ to get $2 = b$, and so the solution to the IVP is $y = \tan^{-1}\left(\frac{x}{2}\right) + 2$.

(b) Use the substitution $u(y(x)) = y'(x)$, so that $u'(y(x))y'(x) = y''(x)$, to solve the IVP $y'' + (y')^2 = 2e^{-y}$ with $y(0) = 0$ and $y'(0) = 2$.

Solution: Make the substitution $y' = u$ so $y'' = u\,u'$. The DE becomes $u\,u' + u^2 = 2e^{-y}$. This is a Bernoulli equation since we can write it as $u' + u = 2e^{-y}\,u^{-1}$. Let $v = u^2$ so $v' = 2u\,u'$. Multiply the Bernoulli equation by $2u$ to get $2u\,u' + 2u^2 = 4e^{-y}$, and write this as $v' + 2v = 4e^{-y}$. This is linear. An integrating factor is $\lambda = e^{\int 2\,dy} = e^{2y}$ and the solution is $v = e^{-2y}\int 4e^y = e^{-2y}(4e^y + b)$. Put in $x = 0$, $y = 0$, $u = y' = 2$ and $v = u^2 = 4$ to get $4 = 4 + b$, so $b = 0$ and we have $v = 4e^{-y}$. Since $v = u^2 = (y')^2$, we have $(y')^2 = 4e^{-y}$ so $y' = \pm 2e^{-y/2}$. Since $y'(0) = 2$ we must use the + sign, so $y' = 2e^{-y/2}$. This DE is separable since we can write it as $e^{y/2}y' = 2$. Integrate both sides to get $2e^{y/2} = 2x + c$. Put in $x = 0$ and $y = 0$ to get $2 = c$, so the solution is given by $2e^{y/2} = 2x + 2$. Solve for $y = y(x)$ to get $y = 2\ln(x+1)$.

3: Use reduction of order to solve each of the following.

(a) Solve the DE $x^3y'' + xy' - y = 0$, given that $y = x$ is one solution.

Solution: Let $y = xu$, $y' = u + xu'$, $y'' = 2u' + xu''$. Put this into the DE to get

$$0 = x^3y'' + xy' - y = 2x^3u' + x^4u'' + xu + x^2u' - xu = x^4u'' + (2x^3 + x^2)u'\,,$$

and so, by dividing by x^4, we get $u'' + \left(\frac{2}{x} + \frac{1}{x^2}\right)u' = 0$. Let $u' = v$ and $u'' = v'$. Then the DE becomes $v' + \left(\frac{2}{x} + \frac{1}{x^2}\right)v = 0$. This is linear. An integrating factor is $\lambda = e^{\int \frac{2}{x} + \frac{1}{x^2}\,dx} = e^{2\ln x - \frac{1}{x}} = x^2e^{-1/x}$ and the solution is $v = \dfrac{e^{1/x}}{x^2}\displaystyle\int 0\,dx = \dfrac{ae^{1/x}}{x^2}$, that is $u' = \dfrac{ae^{1/x}}{x^2}$. Integrate to get $u = \displaystyle\int \frac{ae^{1/x}}{x^2}\,dx = -ae^{1/x} + b$. We only need one more independent solution, so take $a = -1$ and $b = 0$ to get $u = e^{1/x}$. Thus $y = xu = xe^{1/x}$ is a second independent solution. The general solution is $y = Ax + Bxe^{1/x}$.

(b) Solve the IVP $x^2y'' + 3xy' + y = 0$ with $y(1) = 2$, $y'(1) = 3$ given that $y = \frac{1}{x}$ is one solution to the DE.

Solution: Let $y = \frac{1}{x}\,u$ so $y' = -\frac{1}{x^2}\,u + \frac{1}{x}\,u'$ and $y'' = \frac{2}{x^3}\,u - \frac{2}{x^2}\,u' + \frac{1}{x}\,u''$. Put this into the DE to get

$$0 = x^2y'' + 3xy' + y = \tfrac{2}{x}u - 2u' + xu'' - \tfrac{3}{x}u + 3u' + \tfrac{1}{x}u = xu'' + u'\,.$$

Let $u' = v$ and $u'' = v'$, and we get $xv' + v = 0$. This is linear since we can write it as $v' + \frac{1}{x}v = 0$. An integrating factor is $\lambda = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$ and the solution is $v = \frac{1}{x}\int 0 = \frac{a}{x}$, that is $u' = \frac{a}{x}$. Integrate to get $u = a\ln x + b$. We only need one more independent solution so we take $a = 1$, $b = 0$ to get $u = \ln x$. Thus $y = \frac{1}{x}\,u = \frac{\ln x}{x}$ is a second independent solution. The general solution is $y = \frac{A + B\ln x}{x}$. To get $y(1) = 2$ we need $2 = a$, so we have $y = \frac{2 + B\ln x}{x}$. Note that $y' = \frac{B - 2 - B\ln x}{x^2}$, so to get $y'(1) = 3$ we need $3 = B - 2$, so $B = 5$ and the solution to the IVP is $y = \frac{2 + 5\ln x}{x}$.

4: Use variation of parameters to solve each of the following.

(a) Solve the DE $x^2y'' - x(x+2)y' + (x+2)y = 2x^3$ given that $y = x$ and $y = xe^x$ are solutions to the associated homogeneous DE.

Solution: We can write the DE as $y'' - \frac{x+2}{x}\,y' + \frac{x+2}{x^2}\,y = 2x$. Let $r = r(x) = 2x$, $y_1 = y_1(x) = x$ and $y_2 = y_2(x{=}xe^x$ and note that $y_1' = 1$ and $y_2' = (1{+}x)e^x$. A particular solution is given by $y = y_p = y_1u_1{+}y_2u_2$ where $y_1u_1' + y_2u_2' = 0$ and $y_1'u_1' + y_2'u_2' = r$, that is $xu_1' + xe^xu_2' = 0$ (1) and $u_1' + (1+x)e^xu_2' = 2x$ (2). Multiply (1) by $(1+x)$ and multiply (2) by x and subtract to get $x^2u_1' = -2x^2$, that is $u_1' = -2$ so we can take $u_1 = -2x$. Multiply (2) by x and subtract (1) to get $x^2e^xu_2' = 2x^2$, that is $u_2' = 2e^{-x}$, so we can take $u_2 = -2e^{-x}$. A particular solution is given by $y = y_p = y_1u_1{+}y_2u_2 = xu_1{+}xe^xu_2 = -2x^2{-}2x = -2x(x{+}1)$, and so the general solution to the DE is given by $y = Ay_1 + By_2 + y_p = Ax + Bxe^x - 2x(x+1)$.

(b) Solve the DE $xy'' - (1+x)y' + y = x^2e^{2x}$ given that $y = 1+x$ and $y = e^x$ are solutions to the associated homogeneous DE.

Solution: We can write the DE as $y'' - \frac{1+x}{x}\,y' + \frac{1}{x}\,y = xe^{2x}$. Let $r = xe^{2x}$, $y_1 = 1+x$ and $y_2 = e^x$ and note that $y_1' = 1$ and $y_2' = e^x$. To get $y_1u_1' + y_2u_2' = 0$ and $y_1'u_1' + y_+2'u_2' = r$ we need $(1+x)u_1' + e^xu_2' = 0$ (1) and $u_1' + e^xu_2' = xe^{2x}$ (2). Subtract (2) from (1) to get $xu_1' = -xe^{2x}$, that is $u_1' = -e^{2x}$, so we can take $u_1 = -\frac{1}{2}e^{2x}$. Multiply (2) by $(1{+}x)$ and subtract (1) to get $xe^xu_2' = x(1{+}x)e^{2x}$, that is $u_2' = (1{+}x)e^x$, so we can take $u_2 = \int(1{+}x)e^x\,dx = xe^x$. A particular solution is given by $y = y_p = y_1u_1{+}y_2u_2 = (1{+}x)u_1{+}e^xu_2 = -\frac{1}{2}x(1{+}x)e^{2x}{+}xe^{2x} = \frac{1}{2}(x{-}1)e^{2x}$, so the general solution is $y = Ay_1{+}By_2{+}y_p = A(1{+}x){+}Be^x{+}\frac{1}{2}(x{-}1)e^{2x}$.