SYDE Advanced Math 2, Assignment 2 Due Fri May 31, 11:00 pm

1: Consider the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xy \\ x+y \end{pmatrix}$.

(a) In the region $-3 \leq x \leq 3$, $-3 \leq y \leq 3$, sketch the curves $x' = 0$, $y' = 0$, and $\frac{y'}{x'} = \pm\frac{1}{2}, \pm 1 \pm 2$, sketch the direction field field for this system, and sketch the three solution curves through $(1,-1)$, $(1,1)$ and $(-1,1)$.

(b) Use Euler's method with step size $\Delta t = \frac{1}{2}$ to approximate the point $\big(x(2), y(2)\big)$, where $\big(x(t), y(t)\big)$ is the solution to the above system with $\big(x(0), y(0)\big) = (-1, 1)$.

2: Consider the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{y} \\ \frac{2}{x} \end{pmatrix}$.

(a) Solve the system by first solving the DE $\frac{dy}{dx} = \frac{y'}{x'}$ for $y = y(x)$, that is for $y(t) = y(x(t))$.

(b) Solve the system again, this time by eliminating y and y' from x'' to get a second order DE for $x = x(t)$.

(c) Find the unique solution to the system which satisfies the initial conditions $x(0) = 2$ and $y(0) = 1$.

3: Given one solution $\binom{x}{y} = \binom{x(t)}{y(t)}$ to the pair of linear homogeneous ODEs given by $\binom{x'}{y'} = A\binom{x}{y}$, (where A is a 2×2 matrix whose entries are continuous functions of t), we can often find a second independent solution by trying $\binom{x}{y} = \binom{x_2}{y_2} = \begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix}\binom{u}{v} = \binom{u + x_1 v}{y_1 v}$. Then we have $\binom{x'}{y'} = \binom{u' + x_1' v + x_1 v'}{y_1' v + y_1 v'}$ and we have $A\binom{x}{y} = A\binom{u + x_1 v}{y_1 v} = A\binom{u}{0} + A\binom{x_1}{y_1}v = \binom{a_{11}u}{a_{21}u} + \binom{x_1'}{y_1'}v = \binom{a_{11}u + x_1' v}{a_{21}u + y_1' v}$, so the pair of ODEs becomes

$$\binom{0}{0} = \binom{x'}{y'} - A\binom{x}{y} = \binom{u' + x_1' v + x_1 v'}{y_1' v + y_1 v'} - \binom{a_{11}u + x_1' v}{a_{21}u + y_1' v} = \binom{u' + x_1 v' - a_{11}u}{y_1 v_1' - a_{21}u}.$$

Eliminating v' from the two equations $u' + x_1 v' - a_{11}u = 0$ and $y_1 v_1' - a_{21}u = 0$ (by multiplying the first equation by y_1 and the second by x_1 and subtracting) gives $y_1 u' - a_{11} y_1 u + a_{21} x_1 u = 0$ which is a first order linear (and separable) DE for $u = u(x)$. Once we solve for u we have found a second independent solution. Use this method, which is known as **reduction of order**, to solve the IVT given by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{t} \\ \frac{1}{t} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ with $x(1) = 2$ and $y(1) = 3$, given that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{t} \\ -\frac{1}{t} \end{pmatrix}$ is one solution.

4: Given n independent solutions $x_1, x_2, \cdots, x_n$ to the system of homogeneous linear ODEs $x' = Ax$, we can often find a particular solution to the nonhomogeneous system $x' = Ax + b$ by trying $x = x_p = x_1 u_1 + x_2 u_2 + \cdots + x_n u_n = Xu$, where X is the matrix with columns $x_1, \cdots, x_n$ and $u = (u_1, \cdots, u_n)^T$. We then have $x' = X'u + Xu' = AXu + Xu'$ and $Ax = AXu$, so the nonhomogeneous system becomes $b = x' - Ax = AXu + Xu' - AXu = Xu'$, that is $u' = X^{-1}b$. Once we solve for u we have found a particular solution. This method is known as **variation of parameters**.

Use reduction of order and variation of parameters to solve $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{t^2} \\ 1 & \frac{1}{t} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 3\sqrt{t} \end{pmatrix}$ given that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{t} \\ -1 \end{pmatrix}$ is one solution to the associated homogeneous system.