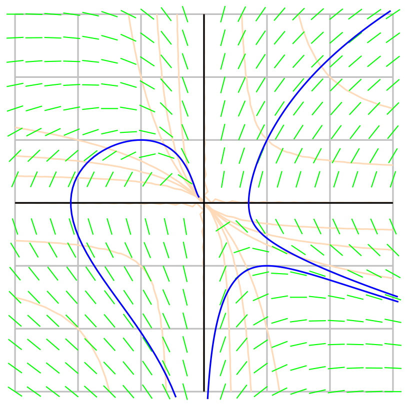
SYDE Advanced Math 2, Solutions to Assignment 2

1: Consider the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xy \\ x+y \end{pmatrix}$.

(a) In the region $-3 \leq x \leq 3$, $-3 \leq y \leq 3$, sketch the curves $x' = 0$, $y' = 0$, and $\frac{y'}{x'} = \pm\frac{1}{2}, \pm 1 \pm 2$, sketch the direction field field for this system, and sketch the three solution curves through $(1,-1)$, $(1,1)$ and $(-1,1)$.

Solution: We have $x' = 0$ when $xy = 0$, so along the x- and y-axes, the slope of any solution curve is vertical. We have $y' = 0$ when $x + y = 0$, so along the line $y = -x$, the slope of any solution curve is zero. The isocline $\frac{dy}{dx} = \frac{y'}{x'} = c$ is given by $\frac{x+y}{xy} = c$, that is $y = \frac{x}{cx-1}$. This is a hyperbola with vertical asymptote along $x = \frac{1}{c}$ and horizontal asymptote along $y = \frac{1}{c}$, and one branch of the hyperbola passes through the origin $(0,0)$. The isoclines $c = \pm\frac{1}{2}, \pm 1, \pm 2$ are shown in peach, the slope field is shown in green, and the solution curves are shown in blue.



(b) Use Euler's method with step size $\Delta t = \frac{1}{2}$ to approximate the point $\big(x(2), y(2)\big)$, where $\big(x(t), y(t)\big)$ is the solution to the above system with $\big(x(0), y(0)\big) = (-1, 1)$.

Solution: We let $t_0 = 0$, $x_0 = -1$, $y_0 = 1$, then set $t_{k+1} = t_k + \Delta t$, $x_{k+1} = x_k + (x_k y_k)\Delta t$ and $y_{k+1} = y_k = (x_k + y_k)\Delta t$. The first few values of t_k, x_k, y_k, $x_k y_k$ and $(x_k + y_k)$ are shown in the table below.

k	t_k	x_k	y_k	$x_k y_k$	$x_k + y_k$
0	0	-1	1	-1	0
1	$\frac{1}{2}$	$-\frac{3}{2}$	1	$-\frac{3}{2}$	$-\frac{1}{2}$
2	1	$-\frac{9}{4}$	$\frac{3}{4}$	$-\frac{27}{16}$	$-\frac{3}{2}$
3	$\frac{3}{2}$	$-\frac{99}{32}$	0	0	$-\frac{99}{32}$
4	2	$-\frac{99}{32}$	$-\frac{99}{64}$		

Thus we have $\big(x(2), y(2)\big) \cong (x_4, y_4) = \big(-\frac{99}{32}, -\frac{99}{64}\big)$.

2: Consider the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{y} \\ \frac{2}{x} \end{pmatrix}$ with initial conditions $x(0) = 2$ and $y(0) = 1$.

(a) Solve the system by first solving the DE $\frac{dy}{dx} = \frac{y'}{x'}$ for $y = y(x)$, that is for $y(t) = y(x(t))$.

Solution: We wish to solve the $\frac{dy}{dx} = \frac{2y}{x}$. This DE is separable since we can write it as $\frac{dy}{y} = \frac{2}{x}\,dx$. Integrate both sides to get $\ln|y| = 2\ln|x| + a$, or equivalently, $y = bx^2$, that is $y(t) = b\,x(t)^2$. Now we return to the system given by $x' = \frac{1}{y}$ and $y' = \frac{2}{x}$. Put $y = bx^2$ into the first DE $x' = \frac{1}{y}$ to get $x' = \frac{1}{bx^2}$. This is separable: we can write it as $x^2 dx = \frac{1}{b} dt$ and integrate to get $\frac{1}{3}x^3 = \frac{1}{b}t + c$, that is $x^3 = \frac{3}{b}t + 3c$. Letting $p = \frac{3}{b}$ and $q = 3c$, we can also write this as $x = (pt+q)^{1/3}$. Finally, we need $y = bx^2 = \frac{3}{p}x^2 = \frac{3}{p}(pt+q)^{2/3}$. Thus the solution is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (pt+q)^{1/3} \\ \frac{3}{p}(pt+q)^{2/3} \end{pmatrix}$$

where $p, q \in \mathbb{R}$.

(b) Solve the system again, this time by eliminating y and y' from x'' to get a second order DE for $x = x(t)$.

Solution: We have pair of DEs, $x' = \frac{1}{y}$ (1) and $y' = \frac{2}{x}$ (2). Differentiate equation (1), then use equations (1) and (2) to get $x'' = -\frac{1}{y^2} \cdot y' = -(x')^2 \cdot \frac{2}{x}$, that is $x\,x'' = 2(x')^2$. Since this second order DE does not involve the variable t, we let $x' = u$ and $x'' = u\,u'$, and the DE becomes $xuu' + 2u^2 = 0$, that is $u' + \frac{2}{x}\,u = 0$. This is linear. An integrating factor is $\lambda = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$ and the solution is $u = \frac{1}{x^2}\int 0\,dx = \frac{a}{x^2}$. Replace u by x' again, and we have the DE $x' = \frac{a}{x^2}$, that is $x^2\,x' = a$. Integrate both sides to get $\frac{1}{3}x^3 = at + b$, that is $x = \big(3at + 3b\big)^{1/3}$. Let $p = 3a$ and $q = 3b$ and rewrite this as $x = \big(pt+q\big)^{1/3}$. Note that $x' = \frac{p}{3}\big(pt+q\big)^{-2/3}$. From equation (1) we have $y = \frac{1}{x'} = \frac{3}{p}\big(pt+q\big)^{2/3}$. Thus the solution to the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \big(pt+q\big)^{1/3} \\ \frac{3}{p}\big(pt+q\big)^{2/3} \end{pmatrix}.$$

(c) Find the unique solution to the system which satisfies the given initial conditions.

Solution: Put $t = 0$, $x = 2$ and $y = 1$ into our solutions $x = \big(pt+q\big)^{1/3}$ and $y = \frac{1}{x'} = \frac{3}{p}\big(pt+q\big)^{2/3}$ to get $2 = q^{1/3}$ and $1 = \frac{3}{p}\,q^{2/3}$, so we must have $q = 8$ and $p = 12$. Thus the solution to the IVP is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \big(12t+8\big)^{1/3} \\ \frac{1}{4}\big(12t+8\big)^{2/3} \end{pmatrix}.$$

3: Use reduction of order to solve the IVT given by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{t} \\ \frac{1}{t} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ with $x(1) = 2$ and $y(1) = 3$, given that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{t} \\ -\frac{1}{t} \end{pmatrix}$ is one solution.

Solution: We try $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$. Putting this into the DE and simplifying gives

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 \\ 0 \end{pmatrix} u = \begin{pmatrix} 1 & 1 + \frac{1}{t} \\ 0 & -\frac{1}{t} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{1}{t} \end{pmatrix} u = \begin{pmatrix} 1 & t+1 \\ 0 & -t \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{t} \end{pmatrix} u = \begin{pmatrix} 1 + \frac{1}{t} \\ -1 \end{pmatrix} u,$$

so we need $u' = (1 + \frac{1}{t})u$ and $v' = -u$. The first ODE is linear as it can be written as $u' - (1 + \frac{1}{t})u = 0$. An integrating factor is $\lambda = e^{\int -(1+\frac{1}{t})dt} = e^{-t-\ln t} = \frac{1}{te^t}$, and the solution is $u = te^t \int 0\, dt = ate^t$. We choose $a = 1$ so that $u = te^t$. The second ODE becomes $v' = -u = -te^t$, so that $v = \int -te^t\, dt$. Integrate by parts to get $v = \int -te^t\, dt = -te^t + \int e^t\, dt = -te^t + e^t + b$. We choose $b = 0$ so that $v = (1-t)e^t$. Thus we obtain a second solution to the system

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 + \frac{1}{t} \\ 0 & -\frac{1}{t} \end{pmatrix} \begin{pmatrix} te^t \\ (1-t)e^t \end{pmatrix} = \begin{pmatrix} te^t + \frac{1}{t}(1-t^2)e^t \\ -\frac{1}{t}(1-t)e^t \end{pmatrix} = \begin{pmatrix} \frac{1}{t}e^t \\ (1-\frac{1}{t})e^t \end{pmatrix}$$

and the general solution to the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + B \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = A \begin{pmatrix} 1 + \frac{1}{t} \\ -\frac{1}{t} \end{pmatrix} + Be^t \begin{pmatrix} \frac{1}{t} \\ 1 - \frac{1}{t} \end{pmatrix}.$$

To get $y(1) = 3$ we need $3 = -A$ so that $A = -3$, and to get $x(1) = 2$ we need $2 = 2A + Be = -6 + Be$ so that $B = 8e^{-1}$. Thus the solution to the IVP is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = -3 \begin{pmatrix} 1 + \frac{1}{t} \\ -\frac{1}{t} \end{pmatrix} + 8e^{t-1} \begin{pmatrix} \frac{1}{t} \\ 1 - \frac{1}{t} \end{pmatrix}.$$

Alternatively we can write this as $x = \frac{8e^{t-1}}{t} - 3(1 + \frac{1}{t})$ and $y = \frac{3}{t} + 8e^{t-1}(1 - \frac{1}{t})$.

4: Use reduction of order and variation of parameters to solve $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{t^2} \\ 1 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 3\sqrt{t} \end{pmatrix}$ given that $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{t} \\ -1 \end{pmatrix}$ is one solution to the associated homogeneous system.

Solution: First we use reduction of order to find a second independent solution to the homogeneous system. We try $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$. Put this into the associated homogeneous DE. and simplify to get

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 \\ 0 \end{pmatrix} u = \begin{pmatrix} 1 & \frac{1}{t} \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = \begin{pmatrix} 1 & \frac{1}{t} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} u,$$

so we need $u' = \frac{1}{t}u$ and $v' = -u$. The first ODE is linear. An integrating factor is $\lambda = e^{\int -\frac{1}{t}\,dt} = e^{-\ln t} = \frac{1}{t}$ and the solution is $u = t\int 0\,dt = at$. The second ODE becomes $v' = -u = -at$, so that $v = -\frac{1}{2}at^2 + b$. We choose $a = 2$ and $b = 0$ so that $u = 2t$ and $v = -t^2$. Thus we obtain the second solution

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1_1^x \\ 0_1^y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{t} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2t \\ -t^2 \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}.$$

Now that we have two independent solutions to the associated homogeneous system, we use variation of parameters to find a particular solution to the given non-homogeneous system. We try $\begin{pmatrix} x_p \\ y_p \end{pmatrix} = X \begin{pmatrix} u \\ v \end{pmatrix}$ where $X = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & t \\ -1 & t^2 \end{pmatrix}$ (and where we are re-using the letters u and v to denote two new functions $u = u(t)$ and $v = v(t)$). Putting this into the given system gives

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = X^{-1} \begin{pmatrix} 4 \\ 3\sqrt{t} \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & t \\ -1 & t^2 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3\sqrt{t} \end{pmatrix} = \frac{1}{2t} \begin{pmatrix} t^2 & -t \\ 1 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} 4 \\ 3\sqrt{t} \end{pmatrix} = \frac{1}{2t} \begin{pmatrix} 4t^2 - 3t\sqrt{t} \\ 4 + \frac{3}{\sqrt{t}} \end{pmatrix}.$$

Since $u' = 2t - \frac{3}{2}\,t^{1/2}$ we have $u = \int 2t - \frac{3}{2}\,t^{1/2}\,dt = t^2 - t^{3/2}$ (plus a constant which we choose to be zero), and since $v' = 2t^{-1} + \frac{3}{2}\,t^{-3/2}$ we have $v = 2\ln t - 3t^{-1/2}$ (plus a constant). Thus we obtain the particular solution

$$\begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & t \\ -1 & t^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{t} & t \\ -1 & t^2 \end{pmatrix} \begin{pmatrix} t^2 - t^{3/2} \\ 2\ln t - 3t^{-1/2} \end{pmatrix} = \begin{pmatrix} t - t^{1/2} + 2t\ln t - 3t^{1/2} \\ -t^2 + t^{3/2} + 2t^2\ln t - 3t^{3/2} \end{pmatrix}.$$

The general solution to the given (non-homogeneous) system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + B\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} x_p \\ y_p \end{pmatrix} = A\begin{pmatrix} \frac{1}{t} \\ -1 \end{pmatrix} + B\begin{pmatrix} t \\ t^2 \end{pmatrix} + \begin{pmatrix} t + 2t\ln t - 4t^{1/2} \\ -t^2 + 2t^2\ln t - 2t^{3/2} \end{pmatrix}.$$