SYDE Advanced Math 2, Assignment 3 Due Fri June 14, 11:00 pm

1: Consider the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

(a) Find the general solution to the system, and find the solution which satisfies $x(0) = 1$ and $y(0) = 1$.

(b) Sketch a phase portrait: show the isocline $x' = 0$ and the isoclines $\frac{y'}{x'} = c$ for $c = 1, 0, -\frac{1}{3}, -\frac{1}{2}, -\frac{3}{5}, -1$, show the direction field, and show the solution curves through each of the six points $(\pm 1, \pm 1)$ and $(0, \pm 1)$.

2: Consider the predator prey model, with the prey species population $x = x(t)$ and the predator species population $y = y(t)$ satisfying the pair of first order ODEs

$$x' = (1 - \tfrac{1}{2}x - \tfrac{1}{2}y)x \quad \text{and} \quad y' = \tfrac{1}{2}(-1 + 2x - y)y\,.$$

Find all the equilibrium points, find the solution to the linearized system at each equilibrium point, and sketch a partial phase portrait which shows the isoclines $x' = 0$ and $y' = 0$ and shows the behaviour of the solution curves near each equilibrium point.

3: Let $x(t)$ be the height of an object of mass m which is thrown upwards from the ground. If the force of air resistance is $-kx'$, then $x(t)$ satisfies the DE $mx'' + kx' + mg = 0$. Suppose that $m = 1$, $k = \frac{1}{10}$ and $g = 10$ so the DE becomes

$$x'' + \tfrac{1}{10}x' + 10 = 0.$$

Letting $x' = u$ and $x'' = u'$ we obtain the equivalent pair of first oder ODEs

$$x' = u \quad \text{and} \quad u' = -\tfrac{1}{10}u - 10.$$

(a) Note that if we let $f(x, u) = u$ and $g(x, u) = -\frac{1}{10}u - 10$, then we do *not* have $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial u} = 0$, so we cannot find a Hamiltonian $H = H(x, u)$ such that $\frac{\partial H}{\partial x} = -g$ and $\frac{\partial H}{\partial y} = f$. However, we can still find a conserved quantity $H = H(x, u)$ as follows: Note that the given second order ODE does not explicitly involve the variable t. Treating u as a function of x with $x' = u$ and $x'' = uu'$, the DE becomes $uu' + \frac{1}{10}u + 10 = 0$. Solve this first order ODE to find $u = u(x)$, and use your solution to find a conserved quantity $H = H(x, u)$ for the given second order ODE (and for the equivalent pair of ODEs).

(b) Sketch the slope field for the pair of first order ODEs in the xu-plane for $0 \le x \le 20$ and $-20 \le u \le 20$. On the same grid, by using a calculator to plot points, accurately sketch the curve $H(x, u) = c$ where $c = H(0, 20)$ (where H is the conserved quantity found in Part (a)).

(c) Given that $x(0) = 0$ and $u(0) = x'(0) = 20$, solve the resulting IVP for $x = x(t)$, find the value of t at which the object reaches its maximum height, and determine whether the object takes longer on the way up to its maximum height, or on the way back down to the ground.

4: An object of mass m, out in space, falls towards the Earth. The force due to gravity is $F = -\frac{GMm}{x^2}$, where x is the distance from the center of the Earth to the object, G is the gravitational constant and M is the mass of the Earth. The position $x = x(t)$ satisfies the second order ODE

$$x'' = -\tfrac{GM}{x^2}.$$

Letting $x' = u$ and $x'' = u'$, we obtain the equivalent pair of first order ODEs

$$x' = u \quad \text{and } u' = -\tfrac{GM}{x^2}.$$

(a) Find a conserved quantity for the second order ODE by applying the method used in the previous question: treating u as a function of x with $x' = u$ and $x'' = u'u$, the second order ODE becomes $uu' = -\frac{GM}{x^2}$. Solve this to find an implicit equation for $u = u(x)$, and hence find a conserved quantity $H = H(x, u)$.

(b) Find a conserved quantity for the pair of ODEs again, this time using the following method: when $f(x, u) = u$ and $g(x, u) = -\frac{GM}{x^2}$, we have $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial u} = 0$. Find $H = H(x, u)$ such that $\frac{\partial H}{\partial x} = -g$ and $\frac{\partial H}{\partial u} = f$.

(c) Given that $x(0) = x_0$ and $x'(0) = 0$, use $H(x, u)$ to find $u = u(x)$, with $u(x) \le 0$ for all $x \ge 0$.

(d) Given that $x(0) = x_0$ and $x'(0) = 0$, use your formula for $u = u(x)$ to find a formula for $t = t(x)$, then find the time at which $x = \frac{1}{2}x_0$. Warning: this involves using substitutions to solve a challenging integral.