SYDE Advanced Math 2, Solutions to Assignment 3

1: Consider the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

(a) Find the general solution to the system, and find the solution which satisfies $x(0) = 1$ and $y(0) = 1$.

Solution: Let $A = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}$. The characteristic polynomial of A is $g(r) = \det A - rI = r^2 - 4r + = (r-2)^2$. The only eigenvalue is $r = 2$. When $r = 2$, we have $A - rI = \begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. The eigenspace is 1-dimensional, and an eigenvector is $u = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. To find a second independent solution, we solve $(A - RI)v = u$. We have

$$(A - rI|u) = \left(\begin{array}{cc|c} 3 & 3 & -1 \\ -3 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{array}\right)$$

so we find that $v = \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix}$. The general solution to the given system is

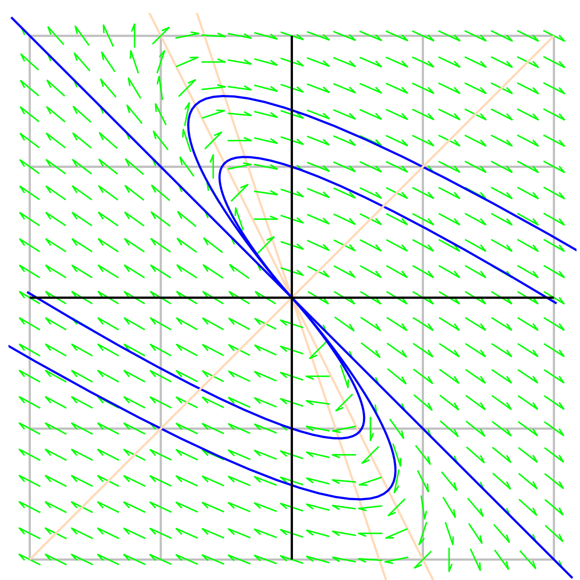
$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + be^{2t} \left(te^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{2t} \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} \right) = e^{2t} \begin{pmatrix} -a - bt - \frac{1}{3}b \\ a + bt \end{pmatrix}.$$

To get $y(0) = 1$ we need $a = 1$ and to get $x(0) = 1$ we need $-a - \frac{1}{3}b = 1$ and so $b = -3(a+1) = -6$. Thus the solution with $x(0) = 1$ and $y(0) = 1$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{2t} \begin{pmatrix} 1 + 6t \\ 1 - 6t \end{pmatrix}$$

(b) Sketch a phase portrait: show the isocline $x' = 0$ and the isoclines $\frac{y'}{x'} = c$ for $c = 1, 0, -\frac{1}{3}, -\frac{1}{2}, -\frac{3}{5}, -1$, show the direction field, and show the solution curves through each of the six points $(\pm 1, \pm 1)$ and $(0, \pm 1)$.

Solution: We have $x' = 0 \iff 5x + 3y = 0$, and $\frac{y'}{x'} = c \iff -3x - y = 5cx + 3cy \iff (3c+1)y = -(5c+3)x$. When $c = -\frac{1}{3}$ this gives $x = 0$ and otherwise it gives $y = -\frac{5c+3}{3c+1}x$. Specifically, when $c = 0$ we get $y = -3x$, when $c = -\frac{1}{2}$ we get $y = x$, when $c = -\frac{3}{5}$ we get $y = 0$ and when $c = -1$ we get $y = -x$. These isoclines are shown in peach, the direction field is shown in green, and the solution curves are shown in blue.



2: Consider the predator prey model, with the prey species population $x = x(t)$ and the predator species population $y = y(t)$ satisfying the pair of first order ODEs

$$x' = (1 - \tfrac{1}{2}x - \tfrac{1}{2}y)x \quad \text{and} \quad y' = \tfrac{1}{2}(-1 + 2x - y)y\,.$$

Find all the equilibrium points, find the solution to the linearized system at each equilibrium point, and sketch a partial phase portrait which shows the isoclines $x' = 0$ and $y' = 0$ and shows the behaviour of the solution curves near each equilibrium point.

Solution: Let $F(x,y) = \Big((1 - \frac{1}{2}x - \frac{1}{2}y)x\,,\, \frac{1}{2}\big(-1 + 2x - y)y\Big)$. The equilibrium points are given by

$$\begin{aligned} F(x,y) = (0,0) &\iff \Big(\big(x = 0 \text{ or } x + y = 2\big) \text{ and } \big(y = 0 \text{ or } 2x - y = 1\big)\Big) \\ &\iff (x,y) = (0,0), (0,1), (2,0) \text{ or } (1,1). \end{aligned}$$

Note that $F(x,y) = \big(x - \frac{1}{2}x^2 - \frac{1}{2}xy\,,\, -\frac{1}{2}y + xy - \frac{1}{2}y^2\big)$ so we have

$$A = \begin{pmatrix} 1 - x - \frac{1}{2}y & -\frac{1}{2}x \\ y & -\frac{1}{2} + x - y \end{pmatrix}.$$

At $(0,0)$ we have $A = DF = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ which has eigenvalues $r = 1, -\frac{1}{2}$ with eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so the general solution to the linearized system at $(0,0)$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + be^{-t/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

At $(0,-1)$ we have $A = \begin{pmatrix} \frac{3}{2} & 0 \\ -1 & \frac{1}{2} \end{pmatrix}$ which has eigenvalues $r = \frac{3}{2}, \frac{1}{2}$. When $r = \frac{1}{2}$ and eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and when $r = \frac{3}{2}$ we have $A - rI = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ so an eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Thus the general solution to the linearized system at $(0,-1)$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^{3t/2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + be^{t/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

At $(2,0)$ we have $A = \begin{pmatrix} -1 & -1 \\ 0 & \frac{3}{2} \end{pmatrix}$ which has eigenvalues $r = -1, \frac{3}{2}$. When $r = -1$ an eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and when $r = \frac{3}{2}$ we have $A = \begin{pmatrix} -\frac{5}{2} & -1 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 5 & 2 \\ 0 & 0 \end{pmatrix}$ so an eigenvector is $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$. The solution to the linearized system at $(2,0)$ is

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + be^{3t/2} \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

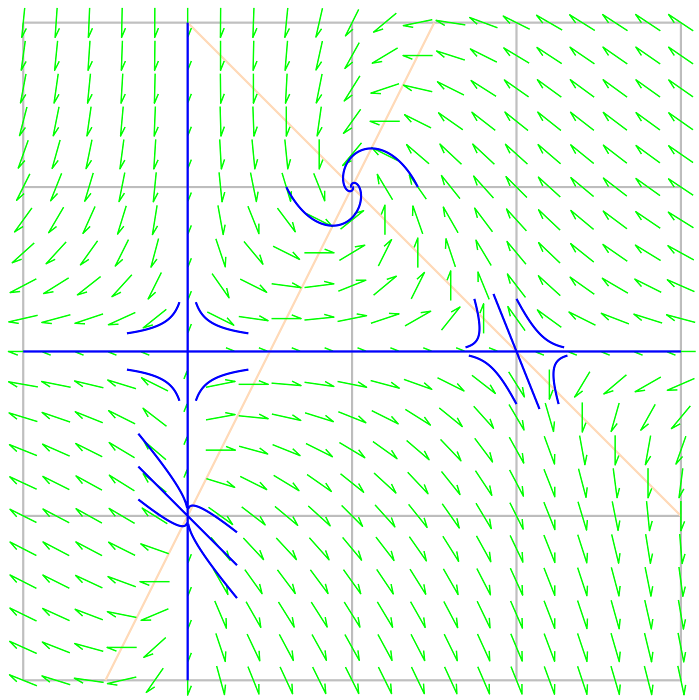
At $(1,1)$ we have $a = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$. The characteristic polynomial is $g(r) = \det(A - rI) = r^2 + r + \frac{3}{4}$. The eigenvalues are $r = \frac{-1 \pm \sqrt{1-3}}{2} = \frac{-1 \pm \sqrt{2}\,i}{2}$. When $r = \frac{-1 + \sqrt{2}\,i}{2}$ we have $A - rI = \begin{pmatrix} -\frac{\sqrt{2}}{2}\,i & -\frac{1}{2} \\ 1 & \frac{\sqrt{2}}{2}\,i \end{pmatrix} \sim \begin{pmatrix} 2 & -\sqrt{2}\,i \\ 0 & 0 \end{pmatrix}$ so an eigenvector is $\begin{pmatrix} \sqrt{2}\,i \\ 2 \end{pmatrix}$. A complex solution is given by

$$e^{(-1+\sqrt{2}\,i)t} \begin{pmatrix} \sqrt{2}\,i \\ 2 \end{pmatrix} = e^{-t/2}\big(\cos \frac{\sqrt{2}\,t}{2} + i \sin \frac{\sqrt{2}\,t}{2}\big) \left(\begin{pmatrix} 0 \\ 2 \end{pmatrix} + i \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \right)$$

and the general real solution is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = ae^{-t/2} \left(\cos \tfrac{\sqrt{2}\,t}{2} \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \sin \tfrac{\sqrt{2}\,t}{2} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \right) + be^{-t/2} \left(\cos \tfrac{\sqrt{2}\,t}{2} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} + \sin \tfrac{\sqrt{2}\,t}{2} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right).$$

Here is a phase portrait: the isoclines $x' = 0$ and $y' = 0$ are shown in pink (except that the solution curves along the axes are in blue), and solutions to the linearized systems at each equilibrium point are shown in blue (these curves approximate the actual solution curves, which are not shown). The question only asks for the portion of the direction field where the solution curves are horizontal or vertical, but we included the full direction field in green (for interest). We remark that the equilibrium point at $(1, 1)$ is attracting (or asymptotically stable), the equilibrium points at $(0, 0)$ and $(2, 0)$ are unstable saddle points, and the equilibrium point at $(0, -1)$ is repelling (this point is not of physical significance since it represents a point where the predator population is negative).



3: Let $x(t)$ be the height of an object of mass m which is thrown upwards from the ground. If the force of air resistance is $-kx'$, then $x(t)$ satisfies the DE $mx''+kx'+mg=0$. Suppose that $m=1$, $k=\frac{1}{10}$ and $g=10$ so the DE becomes

$$x''+\tfrac{1}{10}x'+10=0.$$

Letting $x'=u$ and $x''=u'$ we obtain the equivalent pair of first oder ODEs

$$x'=u \quad\text{and}\quad u'=-\tfrac{1}{10}u-10.$$

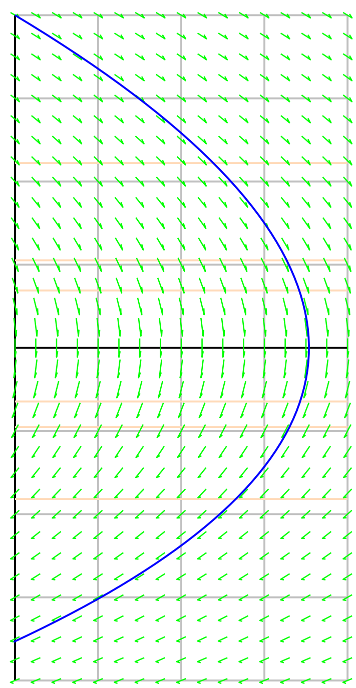
(a) Note that the given second order ODE does not explicitly involve the variable t. Treating u as a function of x with $x'=u$ and $x''=uu'$, the DE becomes $uu'+\frac{1}{10}u+10=0$. Solve this first order ODE to find $u=u(x)$, and use your solution to find a conserved quantity $H=H(x,u)$ for the given second order ODE (and for the equivalent pair of ODEs).

Solution: Letting $u=u(x)$ with $x'=u$ and $x''=uu'$, the DE becomes $uu'+\frac{1}{10}u+10=0$. Dividing by u gives $u'+\frac{1}{10}+\frac{10}{u}=0$, that is $u'=-\frac{u+100}{10u}$. This is a separable DE for $u=u(x)$. We write it in differential form as $\frac{10u}{u+100}\,du=-dx$, that is $\left(10-\frac{1000}{u+100}\right)du=-dx$. Integrate both sides to get $10u-1000\ln(u+100)=-x+c$, that is $x+10u-1000\ln(u+100)=c$. Thus $H(x,u)=x+10u-\ln(u+100)$ is a conserved quantity.

(b) Sketch the slope field for the pair of first order ODEs in the xu-plane for $0\le x\le 20$ and $-20\le u\le 20$. On the same grid, by using a calculator to plot points, accurately sketch the curve $H(x,u)=c$ where $c=H(0,20)$ (where H is the conserved quantity found in Part (a)).

Solution: The solution curves are vertical when $x'=0$, that is when $u=0$, and we have $\frac{u'}{x'}=c$ when $-\frac{1}{10}u-10=cu$, that is when $u=\frac{-10}{c-\frac{1}{10}}=-\frac{100}{10c-1}$. The isoclines $\frac{y'}{x'}=c$ are shown in peach for $c=\pm1,\pm2\pm3$. We have $H(x,u)=x+10u-1000\ln(u+100)$, and $c=H(0,20)=200-1000\ln 120$, so the curve $H(x,u)=c$ is given by $x+10u-1000\ln(u+100)=200-1000\ln 120$, that is $x=200-10u+1000\ln\frac{u+100}{120}$. We make a table of values (for x as a function of u), and plot the curve in blue (notice that it is the curve followed by the solution satisfying the given initial conditions).

u	x
20	0
15	7.44
10	12.99
5	16.47
0	17.68
-5	16.39
-10	12.32
-15	5.16



(c) Given that $x(0) = 0$ and $u(0) = x'(0) = 20$, solve the resulting IVP for $x = x(t)$, find the value of t at which the object reaches its maximum height, and determine whether the object takes longer on the way up to its maximum height, or on the way back down to the ground.

Solution: The given DE $x'' + \frac{1}{10}x' + 10 = 0$ does not involve x, so we let $v = v(t)$ with $x' = v$ and $x'' = v'$. The DE becomes $v' + \frac{1}{10}v = -10$, which is linear for $v = v(t)$. An integrating factor is $\lambda = e^{\int \frac{1}{10}\,dt} = e^{t/10}$, and the solution is

$$v(t) = e^{-t/10}\int -10\,e^{t/10}\,dt = e^{-t/10}\big(-100\,e^{t/10} + b\big) = b\,e^{-t/10} - 100\,.$$

Put in $v(0) = x'(0) = 20$ to get $b - 100 = 20$, so $b = 120$ and we have

$$x'(t) = v(t) = 120\,e^{-t/10} - 100$$
$$x(t) = \int 120\,e^{-t/10} - 100\;dt = -1200\,e^{-t/10} - 100\,t + a$$

Put in $x(0) = 0$ to get $-1200 + a = 0$, so $a = 1200$ and we have

$$x(t) = -1200\,e^{-t/10} - 100\,t + 1200 = 1200\big(1 - e^{-t/10}\big) - 100\,t\,.$$

It reaches its maximum height at height when $v(t) = 0$, and we have

$$v(t) = 0 \Longrightarrow 120\,e^{-t/10} - 100 = 0 \Longrightarrow e^{-t/10} = \tfrac{100}{120} = \tfrac{5}{6} \Longrightarrow e^{t/10} = \tfrac{6}{5} \Longrightarrow \tfrac{1}{10}\,t = \ln\left(\tfrac{6}{5}\right) \Longrightarrow t = 10\ln\left(\tfrac{6}{5}\right).$$

Let $t_1 = 10\ln\frac{6}{5}$, the time at which the height is maximum, and consider its position at $t_2 = 2t_1 = 20\ln\left(\frac{6}{5}\right)$. If it takes longer on the way up, then it will land before $t = t_2$ and then $x(t_2) < 0$. If it takes longer on the way back down, then it will not yet have landed when $t = t_2$ and so we will have $x(t_2) > 0$. We have

$$x(t_2) = 1200\big(1 - e^{-2\ln(6/5)}\big) - 2000\ln\left(\tfrac{6}{5}\right) = 1200\big(1 - \tfrac{25}{36}\big) - 2000\ln\left(\tfrac{6}{5}\right) = 100\big(\tfrac{11}{3} - 20\ln\left(\tfrac{6}{5}\right)\big)\,.$$

A calculator shows that $20\ln\left(\frac{6}{5}\right) \cong 3.64 < \frac{11}{3}$, so $x(t_2) > 0$, and so it takes longer on the way back down.

4: An object of mass m, out in space, falls towards the Earth. The force due to gravity is $F = -\frac{GMm}{x^2}$, where x is the distance from the center of the Earth to the object, G is the gravitational constant and M is the mass of the Earth. The position $x = x(t)$ satisfies the second order ODE

$$x'' = -\tfrac{GM}{x^2}.$$

Letting $x' = u$ and $x'' = u'$, we obtain the equivalent pair of first order ODEs

$$x' = u \quad \text{and } u' = -\tfrac{GM}{x^2}.$$

(a) Find a conserved quantity for these DEs by applying the method used in the previous question: treating u as a function of x with $x' = u$ and $x'' = u'u$, the second order ODE becomes $uu' = -\frac{GM}{x^2}$. Solve this to find an implicit equation for $u = u(x)$, and hence find a conserved quantity $H = H(x, u)$.

Solution: We consider $u = u(x)$ with $x' = u$ and $x'' = u\,u'$. The DE becomes $u\,u' = -\frac{GM}{x^2}$. This DE is separable: we write it as $u\,du = -\frac{GM}{x^2}\,dx$ and integrate both sides to get $\frac{1}{2}\,u^2 = \frac{GM}{x} + c$. Thus we obtain the conserved quantity $H(x, u) = \frac{1}{2}u^2 - \frac{GM}{x}$.

(b) Find a conserved quantity for these DEs again, this time using the following method: when $f(x, u) = u$ and $g(x, u) = -\frac{GM}{x^2}$, we have $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial u} = 0$. Find $H = H(x, u)$ such that $\frac{\partial H}{\partial x} = -g$ and $\frac{\partial H}{\partial u} = f$.

Solution: To get $\frac{\partial H}{\partial x} = -g(x, u) = \frac{1}{10}u + 10$, we need $H = \int \frac{GM}{x^2}\,dx = -\frac{GM}{x} + k$ where $k = k(u)$. Then we have $\frac{\partial H}{\partial u} = k'(u)$, so to get $\frac{\partial H}{\partial u} = f(x, u) = u$, we need $k = \int u\,du = \frac{1}{2}u^2$ (plus a constant, which we choose to be 0). Thus we obtain the conserved quantity $H(x, u) = k(u) - \frac{GM}{x} + k(u) = \frac{1}{2}u^2 - \frac{GM}{x}$.

(c) Given that $x(0) = x_0$ and $x'(0) = 0$, use $H(x, u)$ to find $u = u(x)$, with $u(x) \le 0$ for all $x \ge 0$.

Solution: When $x = x_0$ and $u = 0$, we have $c = H(x, y) = \frac{1}{2}u^2 - \frac{GM}{x} = -\frac{GM}{x_0}$, so the equation $H(x, u) = c$ becomes $\frac{1}{2}\,u^2 = GM\left(\frac{1}{x} - \frac{1}{x_0}\right)$, that is $u = \pm 2\sqrt{2GM}\sqrt{\frac{1}{x} - \frac{1}{x_0}}$. We want $u = u(x) \le 0$, so we must have

$$u = -\sqrt{2GM}\sqrt{\tfrac{1}{x} - \tfrac{1}{x_0}}\,.$$

(d) Given that $x(0) = x_0$ and $x'(0) = 0$, use your formula for $u = u(x)$ to find a formula for $t = t(x)$, then find the time at which $x = \frac{1}{2}x_0$. Warning: this involves using substitutions to solve a challenging integral.

Solution: Replace u by x' again to get $x' = -\sqrt{2GM}\sqrt{\frac{1}{x} - \frac{1}{x_0}}$. This DE is separable as we can write it as $\dfrac{dx}{\sqrt{\frac{1}{x} - \frac{1}{x_0}}} = -\sqrt{2GM}\,dt$. Integrate both sides to get $\displaystyle\int \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{x_0}}} = -\int \sqrt{2GM}\,dt = -\sqrt{2GM}\,t + b_1$. Let I be the integral on the left. Make the substitution $w^2 = \frac{x}{x_0}$ so $\sqrt{x} = \sqrt{x_0}\,w$ and $2\,x_0\,u\,dw = dx$ to get

$$I = \int \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{x_0}}} = \int \frac{\sqrt{x}\,dx}{\sqrt{1 - \frac{x}{x_0}}} = \int \frac{2\,x_0\sqrt{x_0}\,w^2}{\sqrt{1 - w^2}}\,dw.$$

Now let $\cos\theta = w$ so that $\sin\theta = \sqrt{1 - w^2}$ and $-\sin\theta\,d\theta = dw$. Then

$$\begin{aligned} I &= -\int 2\,x_0\sqrt{x_0}\,\cos^2\theta\,d\theta = -x_0\sqrt{x_0}\big(\theta + \sin\theta\cos\theta\big) + b_2 = -x_0\sqrt{x_0}\big(\cos^{-1} u + u\sqrt{1 - u^2}\big) + b_2 \\ &= -x_0\sqrt{x_0}\left(\cos^{-1}\sqrt{\frac{x}{x_0}} + \sqrt{\frac{x}{x_0}}\sqrt{1 - \frac{x}{x_0}}\right) + b_2\,. \end{aligned}$$

Since $I = -\sqrt{2GM}\,t + b_1$, we obtain

$$-x_0\sqrt{x_0}\left(\cos^{-1}\sqrt{\frac{x}{x_0}} + \sqrt{\frac{x}{x_0}}\sqrt{1 - \frac{x}{x_0}}\right) = -\sqrt{2GM}\,t + b$$

where $b = b_1 - b_2$. Put in $t = 0$ and $x = x_0$ to get $b = 0$, and so we have

$$t = \frac{x_0\sqrt{x_0}}{\sqrt{2GM}}\left(\cos^{-1}\sqrt{\frac{x}{x_0}} + \sqrt{\frac{x}{x_0} - \left(\frac{x}{x_0}\right)^2}\right).$$

Finally, when $x = \frac{1}{2}\,x_0$ we have $t = \dfrac{x_0\sqrt{x_0}}{\sqrt{2GM}}\left(\frac{\pi}{4} + \frac{1}{2}\right) = \dfrac{x_0\sqrt{x_0}\,(\pi + 2)}{4\sqrt{2GM}}$.