SYDE Advanced Math 2, Solutions for Practice Problem Set 1

1: (a) Verify that $y = x\sin x$ is a solution of the ODE $y\,(y'' + y) = x\sin 2x$.

Solution: We have $y' = \sin x + x\cos x$ and $y'' = \cos x + \cos x - x\sin x = 2\cos x - x\sin x$ and so

$$\begin{aligned} y\,(y'' + y) &= (x\sin x)(2\cos x - x\sin x + x\sin x) \\ &= (x\sin x)(2\cos x) \\ &= x(2\sin x\cos x) \\ &= x\sin 2x\,. \end{aligned}$$

(b) Find all the solutions of the form $y = ax^2 + bx + c$ to the ODE $\big(y'(x)\big)^2 + 4x = 3y(x) + x^2 + 1$.

Solution: For $y = ax^2 + bx + c$ we have $y' = 2ax + b$, so

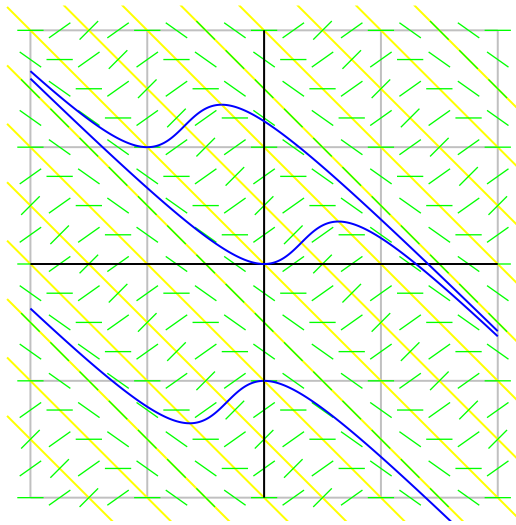
$$\begin{aligned} \big(y'(x)\big)^2 + 4x = 3y(x) + x^2 + 1 \iff& \big(y'(x)\big)^2 + 4x - 3y(x) - x^2 - 1 = 0 \\ \iff& (2ax+b)^2 + 4x - 3(ax^2 + bx + c) - x^2 - 1 = 0 \\ \iff& (4a^2 - 3a - 1)x^2 + (4ab + 4 - 3b)x + (b^2 - 3c - 1) = 0 \\ \iff& 4a^2 - 3a - 1 = 0\ ,\ 4ab + 4 = 3b\ ,\ \text{and } b^2 = 3c + 1 \end{aligned}$$

From $4a^2 - 3a - 1 = 0$ we get $(4a+1)(a-1) = 0$ and so $a = -\frac{1}{4}$ or $a = 1$. When $= -\frac{1}{4}$, the equation $4ab + 4 = 3b$ gives $-1 + 4 = 3b$ so $b = 1$, and then the equation $b^2 = 3c + 1$ gives $1 = 3c + 1$ so $c = 0$. When $a = 1$, $4ab + 4 = 3b$ gives $4b + 4 = 3b$ so $b = -4$ and then $b^2 = 3c + 1$ gives $16 = 3c + 1$ so $c = 5$. Thus there are two solutions, and they are $y = -\frac{1}{4}x^2 + x$ and $y = x^2 - 4x + 5$.

2: Consider the IVP $y' = \sin\big(\pi(x+y)\big)$ with $y(-1) = 1$.

(a) Sketch the direction field for the given ODE for $-2 \le x \le 2$ and $-2 \le y \le 2$ and, on the same grid, sketch the solution curves which pass through each of the points $(-1,1)$, $(0,0)$ and $(0,-1)$.

Solution: We have $y' = 0$ when $\sin\big(\pi(x+y)\big) = 0$, that is when $\pi(x+y) = k\pi$ for some integer k, or equivalently when $x+y = k$ for some integer k. Similarly, we have $y' = 1$ when $x+y = k+\frac{1}{2}$, and $y' = -1$ when $x+y = k-\frac{1}{2}$, and $y' = \frac{1}{2}$ when $x+y = k+\frac{1}{6}$ or $k+\frac{5}{6}$, and $y' = -\frac{1}{2}$ when $x+y = k-\frac{1}{6}$ or $k-\frac{5}{6}$. The isoclines $y' = 0, \pm 1$ are shown in yellow, the direction field is shown in green, and the solution curves are shown in blue.



(b) Using a calculator, apply Euler's method with step size $\Delta x = 0.2$ to approximate the value of $f(0)$ where $y = f(x)$ is the solution to the given IVP.

Solution: We let $x_0 = -1$ and $y_0 = 1$, then for $k \ge 0$ we set $x_{k+1} = x_k + \Delta x$ and $y_{k+1} = y_k + F(x_k, y_k)\Delta x$, where $F(x,y) = \sin\big(\pi(x++y)\big)$. We make a table listing the values of x_k, y_k and $F(x_k, y_k)$.

k	x_k	y_k	$F(x_k, y_k) = x_k - {y_k}^2$
0	-1	1	0
1	-0.8	1	0.5877852524
2	-0.6	1.1117557050	0.9984792328
3	-0.4	1.317252897	0.2570396643
4	-0.2	1.368660830	-0.5054156715
5	0	1.267577696	

Thus we have $f(0) \cong y_5 \cong 1.3$.

3: Solve each of the following ODEs.

(a) $x\,y' + y = \sqrt{x}$.

Solution: This DE is linear since we can write it in the form $y' + \frac{1}{x}\,y = x^{-1/2}$. An integrating factor is $\lambda = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$ and so the solution is $y = \frac{1}{x}\int x\cdot x^{-1/2}\,dx = \frac{1}{x}\int x^{1/2}\,dx = \frac{1}{x}\left(\frac{2}{3}\,x^{3/2} + c\right) = \frac{2}{3}\sqrt{x} + \dfrac{c}{x}$.

(b) $\sqrt{x}\,y' = 1 + y^2$.

Solution: This DE is separable. We can write it as $\dfrac{dy}{1+y^2} = x^{-1/2}\,dx$ and then integrate both sides to get $\tan^{-1} y = 2x^{1/2} + c$, that is $y = \tan\left(2\sqrt{x} + c\right)$.

(c) $y' = x(y^2 - 1)$.

Solution: This DE is separable since (when $y \neq \pm 1$) we can write it as $\dfrac{y'}{y^2 - 1} = x$. Integrate both sides, noting that $\dfrac{1}{y^2-1} = \dfrac{\frac{1}{2}}{y-1} - \dfrac{\frac{1}{2}}{y+1}$, to get

$$
\begin{aligned}
\tfrac{1}{2}\ln\left|\frac{y-1}{y+1}\right| &= \tfrac{1}{2}x^2 + c\ , \text{ where } c \in \mathbb{R} \\
\ln\left|\frac{y-1}{y+1}\right| &= x^2 + 2c \\
\left|\frac{y-1}{y+1}\right| &= e^{x^2+2c} = e^{2c}e^{x^2} \\
\frac{y-1}{y+1} &= \pm e^{2c}e^{x^2} = ae^{x^2}\ , \text{ where } a = \pm e^{2c} \\
y - 1 &= yae^{x^2} + ae^{x^2} \\
y(1 - ae^{x^2}) &= 1 + ae^{x^2} \\
y &= \frac{1 + ae^{x^2}}{1 - ae^{x^2}}\,.
\end{aligned}
$$

Taking $a = 0$ gives the solution $y = 1$, so the general solution is $y = -1$ or $y = \dfrac{1 + ae^{x^2}}{1 - ae^{x^2}}$ with $a \in \mathbb{R}$.

4: Solve each of the following IVPs.

(a) $x\,y' = y^2 + y$ with $y(1) = 1$.

Solution: This DE is separable since we can write it as $\dfrac{y'}{y^2+y} = \dfrac{1}{x}$. Integrate both sides, using partial fractions for the integral on the left, to get

$$\begin{aligned}\int \frac{1}{y} - \frac{1}{y+1}\,dy &= \int \frac{1}{x}\,dx \\ \ln y - \ln(y+1) &= \ln x + c \\ \ln\left(\frac{y}{y+1}\right) &= \ln x + c \\ \frac{y}{y+1} &= e^{\ln x + c} = a\,x\,,\end{aligned}$$

where $a = \ln c$. Put in $y(1) = 1$ to get $\frac{1}{2}$, so we have $\dfrac{y}{y+1} = \dfrac{x}{2}$ so $2y = x(y+1) = xy + x$, that is $y(2-x) = x$, so the solution to the IVP is $y = \dfrac{x}{2-x}$ for $x < 2$.

(b) $x\,y' + 2y = \ln x$ with $y(1) = 0$.

Solution: This DE is linear since we can write it as $y' + \frac{2}{x}\,y = \frac{1}{x}\ln x$. An integrating factor is given by $\lambda = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$ and so the solution is $y = \dfrac{1}{x^2}\displaystyle\int x\,\ln x\,dx$. We integrate by parts using $u = \ln x$ and $dv = x\,dx$ so that $du = \frac{1}{x}\,dx$ and $v = \frac{1}{2}\,x^2$ to get

$$\begin{aligned}y &= \frac{1}{x^2}\int x\,\ln x\,dx \\ &= \frac{1}{x^2}\left(\tfrac{1}{2}x^2\ln x - \int \tfrac{1}{2}\,x\,dx\right) \\ &= \frac{1}{x^2}\left(\tfrac{1}{2}\,x^2\ln x - \tfrac{1}{4}\,x^2 + c\right) \\ &= \frac{c}{x^2} + \tfrac{1}{2}\,\ln x - \tfrac{1}{4}\end{aligned}$$

Put in $y(1) = 0$ to get $0 = c - \frac{1}{4}$, so we have $c = \frac{1}{4}$ and the solution to the IVP is $y = \frac{1}{4}\left(\dfrac{1}{x^2} + 2\ln x - 1\right)$ for $x > 0$.

(c) $y' + xy = x^3$ with $y(0) = 1$.

Solution: This DE is linear. An integrating factor is $\lambda = e^{\int x\,dx} = e^{\frac{1}{2}x^2}$. The solution to the DE is

$$y = e^{-\frac{1}{2}x^2}\int x^3 e^{\frac{1}{2}x^2}\,dx\,.$$

Integrate by parts using $u = x^2$, $du = 2x\,dx$, $v = e^{\frac{1}{2}x^2}$, $dv = xe^{\frac{1}{2}x^2}$ to get

$$y = e^{-\frac{1}{2}x^2}\left(x^2e^{\frac{1}{2}x^2} - \int 2xe^{\frac{1}{2}x^2}\,dx\right) = e^{-\frac{1}{2}x^2}\left(x^2e^{\frac{1}{2}x^2} - 2e^{\frac{1}{2}x^2} + c\right) = x^2 - 2 + ce^{-\frac{1}{2}x^2}\,.$$

To get $y(0) = 1$ we need $-2 + c = 1$ so $c = 3$. Thus the solution to the IVP is

$$y = x^2 - 2 + 3e^{-\frac{1}{2}x^2} \text{ for all } x\,.$$

5: Solve each of the following IVPs.

(a) $y' = \dfrac{x+2}{y-1}$ with $y(1) = -2$.

Solution: This DE is separable since we can write it as $(y-1)y' = (x+2)$. Integrate both sides to get

$$\begin{aligned} \tfrac{1}{2}y^2 - y &= \tfrac{1}{2}x^2 + 2x + c \\ y^2 - 2y &= x^2 + 4x + 2c \\ (y-1)^2 - 1 &= x^2 + 4x + 2c \\ y &= -1 \pm \sqrt{x^2 + 4x + 2c + 1}\,. \end{aligned}$$

To get $y(1) = -2$ we need $1 \pm \sqrt{6 + 2c} = -2$ so we must use the - sign and we must take $c = \frac{3}{2}$. Thus the solution to the IVP is

$$y = 1 - \sqrt{x^2 + 4x + 4} = 1 - \sqrt{(x+2)^2} = 1 - (x+2) = -(x+1) \text{ for } x > -2\,.$$

(b) $y' + y\tan x = \sin^2 x$ with $y(0) = 1$.

Solution: This DE is linear. An integrating factor is $\lambda = e^{\int \tan x\,dx} = e^{\ln(\sec x)} = \sec x = \dfrac{1}{\cos x}$ and the solution to the DE is

$$\begin{aligned} y &= \cos x \int \frac{\sin^2 x}{\cos x}\,dx = \cos x \int \frac{1 - \cos^2 x}{\cos x}\,dx = \cos x \int \sec x - \cos x\;dx \\ &= \cos x\Big(\ln\big|\sec x + \tan x\big| - \sin x + c\Big)\,. \end{aligned}$$

To get $y(0) = 1$ we need $c = 1$, so the solution to the IVP is

$$y = \cos x\Big(\ln\big|\sec x + \tan x\big| - \sin x + 1\Big) \text{ for } -\tfrac{\pi}{2} < x < \tfrac{\pi}{2}\,.$$

(c) $y' = \dfrac{y}{x + y^2}$ with $y(3) = 1$.

Solution: We interchange the rolls of x and y, and solve this DE for $x = x(y)$. We have

$$x'(y) = \frac{1}{y'(x)} = \frac{x + y^2}{y}$$

This DE is linear since we can write it as $x' - \frac{1}{y}\,x = y$. An integrating factor is $\lambda = e^{\int -\frac{1}{y}\,dy} = e^{-\ln y} = \frac{1}{y}$ and the solution is

$$x = y\int 1\,dy = y(y + c) \text{ for } y > 0\,.$$

To get $y(3) = 1$ (that is to get $x(1) = 3$) we need $2 = 1 + c$ so $c = 2$, and so the solution is

$$x = y(y+2) = (y+1)^2 - 1\,.$$

Solve this for $y = y(x)$ to get $y = -1 \pm \sqrt{x+1}$. Note that to satisfy $y(3) = 1$ we need to use the + sign, so

$$y = -1 + \sqrt{x+1} \text{ for } x > 0\,.$$

6: A **Bernoulli** DE is a DE which can be written in the form $y' + py = qy^n$ for some continuous functions p and q and some integer n. The substitution $u = y^{1-n}$ can be used to transform the above Bernoulli DE for $y = y(x)$ into the linear DE $u' + p(1-n)u = q(1-n)$ for $u = u(x)$.

(a) Solve the IVP $y' + y = x\,y^3$, with $y(0) = 2$.

Solution: Let $u = y^{-2}$ so $u' = -2y^{-3}\,y'$, and multiply both sides of the DE $y' + y = x\,y^3$ by $-2y^{-3}$ to get $-2y^{-3}\,y' - 2y^{-2} = -2x$, that is

$$u' - 2u = -2x\,.$$

This is a linear DE for $u = u(x)$. An integrating factor is $I = e^{\int -2\,dx} = e^{-2x}$, and the general solution is $u = e^{2x}\displaystyle\int -2x\,e^{-2x}\,dx$. Integrate by parts using $u = x$, $du = dx$, $v = e^{-2x}$ and $dv = -2e^{-2x}\,dx$ to get

$$u = e^{2x}\left(x\,e^{-2x} - \int e^{-2x}\,dx\right) = e^{2x}\left(x\,e^{-2x} + \tfrac{1}{2}\,e^{-2x} + c\right) = x + \tfrac{1}{2} + c\,e^{2x}\,,$$

that is $y^{-2} = x + \frac{1}{2} + c\,e^{2x}$. To get $y(0) = 2$ we need $\frac{1}{4} = \frac{1}{2} + c$ so $c = -\frac{1}{4}$ and so we have

$$y^{-2} = x + \tfrac{1}{2} - \tfrac{1}{4}\,e^{2x} \Longrightarrow y = \left(x + \tfrac{1}{2} - \tfrac{1}{4}\,e^{2x}\right)^{-1/2} = \frac{2}{\sqrt{4x + 2 - e^{2x}}}\,,$$

for those values of x for which $4x + 2 > e^{2x}$.

(b) Solve the IVP $xy\,y' + y^2 = 1$ with $y(1) = 2$.

Solution: This is a Bernoulli DE since we can write it as $y' + \frac{1}{x}\,y = \frac{1}{x}\,y^{-1}$. We let $u = y^2$ so $u' = 2y\,y'$. Multiply both sides of the DE by $2y$ to get $2y\,y' + \frac{2}{x}\,y^2 = \frac{2}{x}$, that is

$$u' + \tfrac{2}{x}\,u = \tfrac{2}{x}\,.$$

This DE is linear. An integrating factor is $\lambda = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$, and the solution to the DE is

$$u = x^{-2}\int 2x\,dx = x^{-2}\big(x^2 + c\big) = 1 + \frac{c}{x^2}\,,$$

that is $y^2 = 1_{+}\frac{c}{x^2}$, so

$$y = \pm\sqrt{1 + \frac{c}{x^2}}\,.$$

To get $y(1) = 2$ we need $\pm\sqrt{1 + c} = 2$, so we must use the $+$ sign and take $c = 3$. Thus

$$y = \sqrt{1 + \frac{3}{x^2}} = \frac{\sqrt{x^2 + 3}}{x} \text{ for } x > 0\,.$$

7: A **homogeneous** first order DE is a DE which can be written in the form $y' = F\left(\frac{y}{x}\right)$ for some continuous function F. The substitution $u = \frac{y}{x}$ can be used to transform the above homogeneous DE for $y = y(x)$ into the separable DE $xu' = F(u) - u$ for $u = u(x)$.

(a) Solve the IVP $y' = \dfrac{x^2 + 3y^2}{2xy}$ with $y(1) = 2$.

Solution: This DE is homogeneous since we can write it as $y' = \dfrac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$. Let $u = \frac{y}{x}$ so $y = xu$ and $y' = u + xu'$. Then we can write the DE as $u + xu' = \dfrac{1+3u^2}{2u}$, that is $xu' = \dfrac{1+3u^2}{2u} - u = \dfrac{1+u^2}{2u}$. This is separable, as we can write it as $\dfrac{2u\,du}{1+u^2} = \dfrac{dx}{x}$. Integrate both sides to get

$$\ln(1+u^2) = \ln|x| + c \Longrightarrow 1 + u^2 = ax \text{ (where } a = \pm e^c) \Longrightarrow u = \pm\sqrt{ax-1}$$
$$\Longrightarrow \frac{y}{x} = \pm\sqrt{ax-1} \Longrightarrow y = \pm x\sqrt{ax-1}\,.$$

To get $y(1) = 2$, we need $2 = \pm\sqrt{a-1}$, so we need to use the $+$ sign and we need $a - 1 = 4$ so $a = 5$. Thus

$$y = x\sqrt{5x-1} \text{ for } x > \tfrac{1}{5}\,.$$

(b) Solve the IVP $y' = \dfrac{y^2 + 2xy}{x^2}$ with $y(1) = 1$.

Solution: This DE is homogeneous since we can write it as $y' = \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)$. Make the substitution $u = \frac{y}{x}$ so $y = xu$ and $y' = u + xy'$, then the DE becomes $u + xu' = u^2 + u$, that is $xu' = u^2 + 2u$. This new DE is separable since we can write it as $\frac{1}{u^2+u}\,u' = \frac{1}{x}$. Integrate both sides (with respect to x) to get

$$\int \frac{du}{u^2+u} = \int \frac{dx}{x}$$
$$\ln\frac{u}{u+1} = \ln x + a$$
$$\frac{u}{u+1} = bx\,,$$

where $b = e^a$. To get $y(1) = 1$, we put in $x = 1$ and $y = 1$ so $u = \frac{y}{x} = 1$ to get $\frac{1}{2} = b$, thus the solution is given by

$$\frac{u}{u+1} = \frac{x}{2} \Longrightarrow 2u = ux + x \Longrightarrow (2-x)u = x \Longrightarrow u = \frac{x}{2-x} \Longrightarrow \frac{y}{x} = \frac{x}{2-x} \Longrightarrow y = \frac{x^2}{2-x}$$

for $0 < x < 2$.